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Elliptic PDEs, Measures and Capacities

From the Poisson Equation to Nonlinear Thomas–Fermi Problems

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Chapter 1
The Laplacian

"Je parviens à une expression en séries, générale et simple, des attractions des sphéroïdes quelconques très peu différents de la sphère. Il est assez remarquable que cette expression soit donnée sans aucune intégration et par la seule différentiation de fonctions."[1]

Pierre-Simon de Laplace

We present classical properties of harmonic and superharmonic functions, in connection with monotonicity formulas and maximum principles.

1.1 Laplace and Poisson equations

The Laplacian Δ appeared in the 18^{th} and 19^{th} centuries in the studies of gravitational and electrostatic potentials. Assume that $u\colon \Omega \to \mathbb{R}$ denotes a smooth function describing one of these physical quantities in a region where there is no mass or electric charges. Then, the total flux of the field $-\nabla u$ through the boundary of any open ball $B(x;r) \Subset \Omega$ strictly contained in Ω must vanish:

$$\int_{\partial B(x;r)} (-\nabla u) \cdot n \, \mathrm{d}\sigma = 0.$$

The integral above is taken with respect to the surface measure σ, which coincides with the $(N-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$. By the divergence theorem (see Theorem 9.2.4 in [345]), we deduce that, on every ball $B(x;r) \Subset \Omega$, we have

$$-\int_{B(x;r)} \operatorname{div}(\nabla u) = 0,$$

and so u satisfies the *Laplace equation*:

$$\boxed{\Delta u = 0 \;\text{ in } \Omega,}$$

and is called a *harmonic function*.

[1] "I obtain a general and simple expression, in terms of a series, for the attraction of arbitrary spheroids that are not very different from the sphere. It is quite remarkable that this expression is given without any integration and only through the differentiation of functions."

Example 1.1. For every $a \in \mathbb{R}^N$ in dimension $N \geqslant 3$, the function $u\colon \mathbb{R}^N \setminus \{a\} \to \mathbb{R}$ defined for $x \in \mathbb{R}^N \setminus \{a\}$ by

$$u(x) = \frac{1}{|x-a|^{N-2}}$$

is harmonic, which can be checked by differentiating twice this function. An example of radially symmetric harmonic function in dimension $N = 2$ is

$$u(x) = \log \frac{1}{|x-a|}.$$

Exercise 1.1 (radial harmonic functions). Prove that all radial harmonic functions $u\colon \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ in dimension $N \geqslant 3$ are of the form

$$u(x) = \frac{\alpha}{|x|^{N-2}} + \beta,$$

for some $\alpha, \beta \in \mathbb{R}$. What is the counterpart in dimension $N = 2$?

We may also consider the case where u denotes the electrostatic potential generated by some distribution of electric charges of density $\mu\colon \Omega \to \mathbb{R}$. In this case, by Gauss's law the total flux through the boundary of the ball $B(x;r) \Subset \Omega$ equals the total charge inside this ball:

$$\int_{\partial B(x;r)} (-\nabla u) \cdot n \,\mathrm{d}\sigma = \int_{B(x;r)} \mu.$$

By the divergence theorem, we deduce that, for every ball $B(x;r) \Subset \Omega$, we have

$$-\int_{B(x;r)} \operatorname{div}(\nabla u) = \int_{B(x;r)} \mu,$$

and so the function u satisfies the *Poisson equation*:

$$\boxed{-\Delta u = \mu \quad \text{in } \Omega.}$$

A large class of solutions of the Poisson equation is given by the Newtonian potential generated by μ:

Proposition 1.2. *Let $N \geqslant 3$. If $\mu\colon \mathbb{R}^N \to \mathbb{R}$ belongs to the space $C_c^\infty(\mathbb{R}^N)$ of smooth functions with compact support in $\mathbb{R}^N$, then the Newtonian potential $u\colon \mathbb{R}^N \to \mathbb{R}$ defined for $x \in \mathbb{R}^N$ by*

$$u(x) = \frac{1}{(N-2)\sigma_N} \int_{\mathbb{R}^N} \frac{\mu(y)}{|x-y|^{N-2}} \,\mathrm{d}y,$$

where σ_N denotes the measure of the $(N-1)$-dimensional unit sphere $\partial B(0;1)$ in $\mathbb{R}^N$, is a bounded smooth function satisfying the Poisson equation $-\Delta u = \mu$ in $\mathbb{R}^N$.

The function $F\colon \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ defined by

$$F(z) = \frac{1}{(N-2)\sigma_N} \frac{1}{|z|^{N-2}}$$

is called the *fundamental solution* of the Laplacian in dimension $N \geqslant 3$. Using this notation, we write the Newtonian potential as a *convolution* between F and μ:

$$u(x) = \int_{\mathbb{R}^N} F(x-y)\mu(y)\,\mathrm{d}y = (F * \mu)(x).$$

Proof of Proposition 1.2. Since the density μ has compact support and F is locally integrable, u is well defined. By the change of variables $z = x - y$, we have

$$u(x) = \int_{\mathbb{R}^N} F(z)\mu(x-z)\,\mathrm{d}z,$$

whence the function u is smooth. Differentiating under the integral sign, and then undoing the previous change of variables, we get

$$\Delta u(x) = \int_{\mathbb{R}^N} F(x-y)\,\Delta\mu(y)\,\mathrm{d}y. \tag{1.1}$$

We now identify the integrand as the divergence of a smooth vector field in $\mathbb{R}^N \setminus \{x\}$. For this purpose, denote $F_x(y) = F(x-y) = F(y-x)$. We have

$$\begin{aligned} \operatorname{div}(F_x\,\nabla\mu) &= \nabla F_x \cdot \nabla\mu + F_x\,\Delta\mu, \\ \operatorname{div}(\mu\,\nabla F_x) &= \nabla\mu \cdot \nabla F_x + \mu\,\Delta F_x. \end{aligned}$$

Since the function F_x is harmonic in $\mathbb{R}^N \setminus \{x\}$ (Example 1.1), subtracting the second identity from the first one we deduce that

$$\operatorname{div}(F_x\,\nabla\mu - \mu\,\nabla F_x) = F_x\,\Delta\mu.$$

The function $F_x\,\nabla\mu - \mu\,\nabla F_x$ has compact support in $\mathbb{R}^N$ and is smooth on $\mathbb{R}^N \setminus \{x\}$. We are thus allowed to apply the divergence theorem to open sets of the form $\mathbb{R}^N \setminus B[x;r]$. For every $r > 0$, we then get

$$\int_{\mathbb{R}^N \setminus B[x;r]} F_x(y)\,\Delta\mu(y)\,\mathrm{d}y = \int_{\partial B(x;r)} (F_x\,\nabla\mu - \mu\,\nabla F_x)\cdot(-n)\,\mathrm{d}\sigma, \tag{1.2}$$

where $n(z) = (z-x)/r$ is the outward normal vector with respect to $B(x;r)$.

For every $z \in \partial B(x;r)$, we now compute

$$F_x(z) = \frac{1}{(N-2)\sigma_N r^{N-2}}$$

and

$$\nabla F_x(z) \cdot n(z) = -\frac{1}{\sigma_N r^{N-1}}.$$

Rewriting the right-hand side of identity (1.2) in terms of average integrals over spheres, we get

$$\int_{\mathbb{R}^N \setminus B[x;r]} F_x(y)\, \Delta\mu(y)\, \mathrm{d}y = -\frac{r}{N-2} \fint_{\partial B(x;r)} \nabla\mu \cdot n \, \mathrm{d}\sigma - \fint_{\partial B(x;r)} \mu \, \mathrm{d}\sigma.$$

Since the right-hand side converges to $-\mu(x)$ as $r \to 0$, by identity (1.1) and the dominated convergence theorem we deduce that

$$\Delta u(x) = \lim_{r \to 0} \int_{\mathbb{R}^N \setminus B[x;r]} F_x(y)\, \Delta\mu(y)\, \mathrm{d}y = -\mu(x). \qquad \square$$

According to the proposition above, for every function $\mu \in C_c^\infty(\mathbb{R}^N)$ we have

$$-\Delta(F * \mu) = \mu \ \text{ in } \mathbb{R}^N.$$

By an affine change of variable and by differentiation under the integral sign, we obtain the following representation formula involving the Laplacian: for every $x \in \mathbb{R}^N$,

$$\mu(x) = -\int_{\mathbb{R}^N} F(x-y)\, \Delta\mu(y)\, \mathrm{d}y.$$

Exercise 1.2 (representation formulas in dimensions 1 and 2). Prove that

(a) given $\mu \in C_c^\infty(\mathbb{R})$, for every $x \in \mathbb{R}$ we have

$$\mu(x) = \frac{1}{2} \int_{\mathbb{R}} |x-y|\, \mu''(y)\, \mathrm{d}y;$$

(b) given $\mu \in C_c^\infty(\mathbb{R}^2)$, for every $x \in \mathbb{R}^2$ we have

$$\mu(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y|\, \Delta\mu(y)\, \mathrm{d}y.$$

The Newtonian potential satisfies the Poisson equation when the density μ is merely a Hölder-continuous function of exponent α for some $0 < \alpha < 1$, but the proof of the counterpart of Proposition 1.2 has to be substantially modified, see Lemma 4.2 in [146]. In this case, the Newtonian potential is a $C^{2,\alpha}$ function.

Chapter 2
Poisson equation

> "Les fonctions de domaine ont un sens physique très clair : ce sont les nombres qui mesurent des grandeurs. À cet égard, ces nombres s'introduisent en physique plus primitivement même que les fonctions de point, lesquelles ne servent le plus souvent qu'à étalonner des qualités."[1]
>
> Henri Lebesgue

We now consider solutions of the Poisson equation

$$-\Delta u = \mu$$

in the sense of distributions. As a consequence of the Riesz representation theorem, every weakly superhamonic function satisfies the Poisson equation for some nonnegative Borel measure μ.

2.1 Finite measures

We briefly recall in this section the definition and some properties of finite Borel measures.

Definition 2.1. Given a measure space X equipped with a σ-algebra Σ, a *finite measure* ν is a set function $\nu\colon \Sigma \to \mathbb{R}$ such that, for every sequence $(A_n)_{n\in\mathbb{N}}$ of disjoint subsets belonging to Σ, we have

$$\nu\Big(\bigcup_{k=0}^{\infty} A_k\Big) = \sum_{k=0}^{\infty} \nu(A_k).$$

In particular, we have $\nu(\emptyset) = 0$. Measures are natural objects in gravitational and electrostatic problems, since $\nu(A)$ can be physically interpreted as the mass or electric charge contained in the set A.

[1] "Set functions have a clear physical meaning: these are the numbers that measure quantities. In this respect, these numbers are introduced in physics more primitively than point functions, which are most often used to calibrate properties."

Exercise 2.1 (monotone set lemma). Let ν be a finite measure on a measure space X. Prove that

(*a*) if $(A_n)_{n\in\mathbb{N}}$ is a *nondecreasing* sequence of sets in Σ, then

$$\nu\Big(\bigcup_{k=0}^{\infty} A_k\Big) = \lim_{n\to\infty} \nu(A_n);$$

(*b*) if $(A_n)_{n\in\mathbb{N}}$ is a *non-increasing* sequence of sets in Σ, then

$$\nu\Big(\bigcap_{k=0}^{\infty} A_k\Big) = \lim_{n\to\infty} \nu(A_n).$$

In analogy with Definition 2.1, one also considers nonnegative measures ν, defined in the σ-algebra Σ, taking values in the interval $[0, +\infty]$. An important example is the Lebesgue measure on $\mathbb{R}^N$.

Exercise 2.2. Let ν be a nonnegative measure on a measure space X. Prove that

(*a*) ν is *monotone*: if $A, B \in \Sigma$ and $A \subset B$, then $\nu(A) \leqslant \nu(B)$;

(*b*) ν is *subadditive*: if $(A_n)_{n\in\mathbb{N}}$ is a sequence of elements in Σ, not necessarily disjoint, then

$$\nu\big(\bigcup_{k=0}^{\infty} A_k\big) \leqslant \sum_{k=0}^{\infty} \nu(A_k).$$

If X is a locally compact topological space, an important example of σ-algebra is the class of Borel subsets of X: this is the smallest σ-algebra containing all compact – or equivalently all open – subsets of X. The typical example of locally compact space X we consider is given by an open subset $\Omega \subset \mathbb{R}^N$ endowed with the Euclidean topology. We then denote by $\mathcal{M}(\Omega)$ the vector space of finite Borel measures on Ω, and we equip this space with the *total variation norm* defined by

$$\boxed{\|\mu\|_{\mathcal{M}(\Omega)} = \sup\,\{\mu(A) - \mu(B) \colon A, B \in \mathcal{B}(\Omega)\}.}$$

Exercise 2.3. Prove that $\|\cdot\|_{\mathcal{M}(\Omega)}$ is a norm in $\mathcal{M}(\Omega)$.

One of the reasons for choosing such a norm is to recover the usual L^1 norm when dealing with densities of the Lebesgue measure:

Example 2.2. For every summable function $f \colon \Omega \to \mathbb{R}$, denote by λ_f the measure given in terms of the Lebesgue measure with density f: for every Borel set $A \subset \Omega$,

$$\lambda_f(A) = \int_A f.$$

By definition of the integral, this measure λ_f satisfies

$$\int_\Omega \psi \, \mathrm{d}\lambda_f = \int_\Omega \psi f$$

for every measurable step function $\psi\colon \Omega \to \mathbb{R}$, and so for every bounded measurable function ψ. The supremum in the definition of the total variation norm is achieved using the sets $A = \{f > 0\}$ and $B = \{f < 0\}$, and we have

$$\|\lambda_f\|_{\mathcal{M}(\Omega)} = \int_{\{f>0\}} f - \int_{\{f<0\}} f = \|f\|_{L^1(\Omega)}.$$

Throughout the book, we identify the measure λ_f with its associated density f.

A convenient characterization of the total variation norm is provided by the positive and negative parts of the measure:

$$\boxed{\|\mu\|_{\mathcal{M}(\Omega)} = \mu^+(\Omega) + \mu^-(\Omega) = |\mu|(\Omega).} \tag{2.1}$$

The definitions of the measures μ^+ and μ^- are based on the Jordan decomposition theorem, see Theorem 8.2 in [21] or Theorem 3.3 [134]:

Proposition 2.3. *Let $(X; \nu)$ be a measure space. Then, there exists a measurable set $E \subset X$ such that*

(I) *for every measurable set $A \subset E$, we have $\nu(A) \geqslant 0$,*

(II) *for every measurable set $A \subset X \setminus E$, we have $\nu(A) \leqslant 0$.*

Denoting by $E \subset \Omega$ any Borel set given by the Jordan decomposition theorem, the *positive part of* μ is the nonnegative measure μ^+ defined for every Borel set $A \subset \Omega$ by contraction of μ on E as

$$\mu^+(A) = \mu\lfloor_E(A) = \mu(A \cap E),$$

and the *negative part of* μ is the nonnegative measure μ^- defined for every Borel set $A \subset \Omega$ by contraction of $-\mu$ on $\Omega \setminus E$ as

$$\mu^-(A) = -\mu\lfloor_{\Omega\setminus E}(A) = -\mu(A \setminus E).$$

The notations $\max\{\mu, 0\}$ to denote the measure μ^+ and $\min\{\mu, 0\}$ for $-\mu^-$ are also used. By the additivity of μ, we have

$$\mu = \mu^+ - \mu^- = \max\{\mu, 0\} + \min\{\mu, 0\},$$

and the *total variation measure* $|\mu|$ is defined as

$$|\mu| = \mu^+ + \mu^-.$$

Exercise 2.4. Prove identity (2.1).

The vector space of finite Borel measures $\mathcal{M}(\Omega)$ equipped with the total variation norm is a Banach space, but is not separable since, for every distinct points $a, b \in \Omega$, the Dirac mass gives

$$\|\delta_a - \delta_b\|_{\mathcal{M}(\Omega)} = \delta_a(\Omega) + \delta_b(\Omega) = 2.$$

Here, the Dirac mass δ_a is the measure defined for every Borel set $A \subset \mathbb{R}^N$ by

$$\delta_a(A) = \begin{cases} 1 & \text{if } a \in A, \\ 0 & \text{if } a \notin A. \end{cases}$$

One may try to approximate a given finite measure μ in Ω using some sequence of measures $(\mu_n)_{n\in\mathbb{N}}$ having better properties like density or capacitary upper bounds (cf. Chapter 14). In general, this is very difficult to achieve – or simply impossible – using the strong convergence with respect to the total variation norm:

$$\lim_{n\to\infty} \|\mu_n - \mu\|_{\mathcal{M}(\Omega)} = 0.$$

For instance, the Dirac mass δ_a with $a \in \Omega$ cannot be strongly approximated by summable functions since, for every $f \in L^1(\Omega)$, we have

$$\|f - \delta_a\|_{\mathcal{M}(\Omega)} = \|f\|_{L^1(\Omega)} + 1.$$

Behind this obstruction lies a more general fact: singular measures cannot be strongly approximated by absolutely continuous measures.

Exercise 2.5. Prove that if $(f_n)_{n\in\mathbb{N}}$ is a sequence in $L^1(\Omega)$ converging strongly to some measure μ in $\mathcal{M}(\Omega)$, then $\mu = f$ for some summable function f.

In many situations it suffices to have convergence in the weak sense, sometimes also called *vague* convergence:

Definition 2.4. Let $\mu \in \mathcal{M}(\Omega)$. A sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}(\Omega)$ converges weakly to μ in the sense of measures if, for every $\phi \in C_c^0(\Omega)$, we have

$$\lim_{n\to\infty} \int_\Omega \phi \, d\mu_n = \int_\Omega \phi \, d\mu,$$

where $C_c^0(\Omega)$ denotes the set of continuous functions with compact support in Ω.

Another characterization of the total variation norm in connection with this notion of convergence is the following:

$$\boxed{\|\mu\|_{\mathcal{M}(\Omega)} = \sup\left\{\int_\Omega \phi \,\mathrm{d}\mu : \phi \in C_c^0(\Omega) \text{ and } |\phi| \leqslant 1 \text{ in } \Omega\right\}.} \tag{2.2}$$

This identity is based on the following remarkable regularity property of measures, discovered by Lebesgue in $\mathbb{R}^N$, see Section 7.2 in [134]:

Proposition 2.5. *Let X be a locally compact space, and let ν be a finite Borel measure on X. Then, for every Borel set $A \subset \Omega$ and every $\epsilon > 0$, we have*

(I) ***inner regularity**: there exists a compact set $K \subset A$ such that $|\nu(A \setminus K)| \leqslant \epsilon$,*

(II) ***outer regularity**: there exists an open set $U \supset A$ such that $|\nu(U \setminus A)| \leqslant \epsilon$.*

According to this proposition, a finite measure is determined by its values on all compact or all open subsets.

Exercise 2.6. Let $\mu \in \mathcal{M}(\Omega)$.

(a) Prove that

$$\|\mu\|_{\mathcal{M}(\Omega)} = \sup\{\mu(K) - \mu(L) : K, L \subset \Omega \text{ are compact and disjoint}\}.$$

(b) Deduce identity (2.2).

The total variation norm is lower semicontinuous with respect to the weak convergence:

Proposition 2.6. *If $(\mu_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{M}(\Omega)$ converging weakly to μ in the sense of measures on Ω, then*

$$\|\mu\|_{\mathcal{M}(\Omega)} \leqslant \liminf_{n\to\infty} \|\mu_n\|_{\mathcal{M}(\Omega)}.$$

Proof. For every $\phi \in C_c^0(\Omega)$ such that $|\phi| \leqslant 1$ in Ω, we have

$$\left|\int_\Omega \phi \,\mathrm{d}\mu_n\right| \leqslant \|\mu_n\|_{\mathcal{M}(\Omega)}.$$

Taking the limit as $n \to \infty$, the weak convergence of the sequence $(\mu_n)_{n\in\mathbb{N}}$ yields that

$$\int_\Omega \phi \,\mathrm{d}\mu \leqslant \liminf_{n\to\infty} \|\mu_n\|_{\mathcal{M}(\Omega)}.$$

The estimate follows from the characterization (2.2) by taking the supremum of the left-hand side with respect to ϕ. □

We now prove that every finite Borel measure can be approximated by smooth functions in the weak sense:

Proposition 2.7. *For every $\mu \in \mathcal{M}(\Omega)$, there exists a sequence of summable functions $(f_n)_{n\in\mathbb{N}}$ in $C^\infty(\overline{\Omega})$ converging weakly to μ in the sense of measures on Ω, and such that*

$$\lim_{n\to\infty} \|f_n\|_{L^1(\Omega)} = \|\mu\|_{\mathcal{M}(\Omega)}.$$

The argument is based on a convolution of μ using a sequence of *mollifiers* $(\rho_n)_{n\in\mathbb{N}}$: for every $n \in \mathbb{N}$, $\rho_n \in C_c^\infty(\mathbb{R}^N)$ is a nonnegative function such that

$$\int_{\mathbb{R}^N} \rho_n = 1$$

and, for every $\eta > 0$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N\setminus B(0;\eta)} \rho_n = 0.$$

A convenient choice of mollifiers consists in taking

$$\rho_n(x) = \frac{1}{\epsilon_n^N}\rho\Big(\frac{x}{\epsilon_n}\Big)$$

for some fixed function $\rho \in C_c^\infty(\mathbb{R}^N)$ and some sequence of positive numbers $(\epsilon_n)_{n\in\mathbb{N}}$ converging to zero.

Proof of Proposition 2.7. Given a sequence of mollifiers $(\rho_n)_{n\in\mathbb{N}}$, for every $n \in \mathbb{N}$ let $\rho_n * \mu: \mathbb{R}^N \to \mathbb{R}$ be the convolution defined for $x \in \mathbb{R}^N$ as

$$\rho_n * \mu(x) = \int_\Omega \rho_n(x-y)\,\mathrm{d}\mu(y).$$

In particular, $\rho_n * \mu \in C^\infty(\overline{\Omega})$, and we also have $\rho_n * \mu \in L^1(\mathbb{R}^N)$. If in addition ρ_n is an even function, then, for every $\phi \in C_c^0(\Omega)$, it follows from Fubini's theorem that

$$\int_\Omega \phi\, \rho_n * \mu = \int_\Omega \bigg(\int_\Omega \rho_n(x-y)\phi(x)\,\mathrm{d}x\bigg)\,\mathrm{d}\mu(y) = \int_\Omega \rho_n * \phi\,\mathrm{d}\mu.$$

Since $\phi \in C_c^0(\Omega)$, the sequence $(\rho_n * \phi)_{n\in\mathbb{N}}$ converges uniformly to ϕ in $\overline{\Omega}$, from which we deduce the weak convergence of $(\rho_n * \mu)_{n\in\mathbb{N}}$ to μ in the sense of measures.

To conclude with $f_n = \rho_n * \mu$, by the lower semicontinuity of the norm under weak convergence (Proposition 2.6) it suffices to check that, for every $n \in \mathbb{N}$, we have

$$\|\rho_n * \mu\|_{L^1(\Omega)} \leqslant \|\mu\|_{\mathcal{M}(\Omega)}.$$

This is a consequence of Fubini's theorem. Indeed, for every $x \in \Omega$,

$$|\rho_n * \mu(x)| \leqslant \int_\Omega \rho_n(x-y)\,\mathrm{d}|\mu|(y).$$

Thus, by Fubini's theorem, we get

$$\|\rho_n * \mu\|_{L^1(\Omega)} \leqslant \int_\Omega \left(\int_\Omega \rho_n(x-y)\,\mathrm{d}x\right)\mathrm{d}|\mu|(y) \leqslant \int_\Omega \mathrm{d}|\mu|(y) = \|\mu\|_{\mathcal{M}(\Omega)},$$

and the conclusion follows. □

We close this section with the weak compactness property of bounded sequences of measures due to Radon, see p. 1337 in [290]:

Proposition 2.8. *If $(\mu_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\mathcal{M}(\Omega)$, then there exists a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ converging weakly to some $\mu \in \mathcal{M}(\Omega)$ in the sense of measures on Ω.*

The main ingredient in the proof of Proposition 2.8 is the Riesz representation theorem that was originally stated by F. Riesz [294] as a way of identifying a continuous linear functional in the space of continuous functions, see Section 7.1 in [134]:

Proposition 2.9. *Let X be a locally compact metric space. If $T\colon C_c^0(X) \to \mathbb{R}$ is a linear functional such that, for every $\phi \in C_c^0(X)$,*

$$|T(\phi)| \leqslant C \sup_X |\phi|,$$

then there exists a unique finite Borel measure ν on X such that, for every $\phi \in C_c^0(X)$,

$$T(\phi) = \int_X \phi\,\mathrm{d}\nu.$$

Proof of Proposition 2.8. Let D be a countable subset of $C_c^0(\Omega)$. Using a diagonalization argument, there exists a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ such that, for every $\phi \in D$, the limit

$$\lim_{k\to\infty} \int_\Omega \phi\,\mathrm{d}\mu_{n_k}$$

exists. Choosing D to be a dense subset of $C_c^0(\Omega)$ with respect to the sup norm, it follows that such a limit exists for every $\phi \in C_c^0(\Omega)$, and we define

$$T(\phi) = \lim_{k\to\infty} \int_\Omega \phi\,\mathrm{d}\mu_{n_k}.$$

Since

$$\left|\int_\Omega \phi\,\mathrm{d}\mu_{n_k}\right| \leqslant \|\mu_{n_k}\|_{\mathcal{M}(\Omega)} \sup_\Omega |\phi|,$$

for every $k \in \mathbb{N}$, the functional $T: C_c^0(\Omega) \to \mathbb{R}$ satisfies

$$|T(\phi)| \leqslant (\liminf_{k\to\infty} \|\mu_{n_k}\|_{\mathcal{M}(\Omega)}) \sup_{\Omega} |\phi|.$$

Hence, by the Riesz representation theorem, there exists a finite measure μ on Ω such that, for every $\phi \in C_c^0(\Omega)$,

$$\lim_{k\to\infty} \int_\Omega \phi \, \mathrm{d}\mu_{n_k} = T(\phi) = \int_\Omega \phi \, \mathrm{d}\mu.$$

This gives the conclusion. □

2.2 Distributional solutions

We are interested in solutions of the Poisson equation for some measure data:

Definition 2.10. Let $\mu \in \mathcal{M}(\Omega)$. We say that u is a solution of the *Poisson equation*

$$-\Delta u = \mu$$

in the sense of distributions in Ω if $u \in L^1_{\mathrm{loc}}(\Omega)$ and if u satisfies, for every $\varphi \in C_c^\infty(\Omega)$, the integral identity

$$-\int_\Omega u \, \Delta\varphi = \int_\Omega \varphi \, \mathrm{d}\mu.$$

If $u: \Omega \to \mathbb{R}$ is a smooth function, then

$$\operatorname{div}(u\nabla\varphi - \varphi\nabla u) = u \, \Delta\varphi - \varphi \, \Delta u.$$

Hence, by the divergence theorem we have that

$$-\int_\Omega u \, \Delta\varphi = \int_\Omega \varphi(-\Delta u),$$

for every $\varphi \in C_c^\infty(\Omega)$.

We now consider two examples inducing legitimate measures, that rely on the representation formula (Proposition 1.2): for every $\varphi \in C_c^\infty(\mathbb{R}^N)$,

$$\varphi = -\Delta(F * \varphi) = -F * (\Delta\varphi), \tag{2.3}$$

where F is the fundamental solution of the Laplacian.

Example 2.11. For every $a \in \mathbb{R}^N$, the function

$$F_a : \mathbb{R}^N \longrightarrow \mathbb{R}$$

defined for $y \in \mathbb{R}^N \setminus \{a\}$ by

$$F_a(y) = F(y - a)$$

satisfies the Poisson equation

$$-\Delta F_a = \delta_a$$

in the sense of distributions in $\mathbb{R}^N$. Indeed, by the definition of the convolution product and the representation formula (2.3), for every $\varphi \in C_c^\infty(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} F_a \, \Delta\varphi = (F * \Delta\varphi)(a) = -\varphi(a) = -\int_{\mathbb{R}^N} \varphi \, \mathrm{d}\delta_a .$$

Example 2.12. Let $N \geqslant 3$. The *Newtonian potential* generated by a nonnegative measure $\mu \in \mathcal{M}(\mathbb{R}^N)$ is the function

$$\mathcal{N}\mu : \mathbb{R}^N \longrightarrow [0, +\infty]$$

defined by

$$\mathcal{N}\mu(x) = \int_{\mathbb{R}^N} F(x - y) \, \mathrm{d}\mu(y) = \frac{1}{(N-2)\sigma_N} \int_{\mathbb{R}^N} \frac{\mathrm{d}\mu(y)}{|x - y|^{N-2}} .$$

We note that $\mathcal{N}\mu$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and satisfies

$$-\Delta \mathcal{N}\mu = \mu$$

in the sense of distributions in $\mathbb{R}^N$. Indeed, by Fubini's theorem and by the representation formula (2.3), for every $\varphi \in C_c^\infty(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} \mathcal{N}\mu \, \Delta\varphi = \int_{\mathbb{R}^N} (F * \Delta\varphi) \, \mathrm{d}\mu = -\int_{\mathbb{R}^N} \varphi \, \mathrm{d}\mu .$$

The original approach in the 1920s and 1930s to investigate the Poisson equation with measure data using the Newtonian potential was later superseded by the formulation in the sense of distributions in the 1940s thanks to the flexibility and greater generality of the latter. We now verify that every solution of the Poisson equation with nonnegative measure datum equals the Newtonian potential modulo a harmonic function:

Proposition 2.13. *Let $N \geqslant 3$, and let $\mu \in \mathcal{M}(\Omega)$ be a nonnegative measure. If $u \in L^1_{\mathrm{loc}}(\Omega)$ is a solution of the Poisson equation with density μ, then there exists a harmonic function $h : \Omega \to \mathbb{R}$ such that*

$$u = \mathcal{N}\mu + h$$

almost everywhere in Ω.

Chapter 4
Variational approach

> "... le condizioni particolarissime in cui l'Hilbert tratta il problema di Dirichlet appaiono elemento integrante delle sue deduzioni e pare lascino ben poca speranza che con ragionamenti analoghi possa trattarsi, senza profonde modificazioni, il problema generale."[1]
>
> Beppo Levi

We prove the existence of variational solutions of the Dirichlet problem

$$\begin{cases} -\Delta u + g(u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

when the nonlinearity g satisfies the sign condition and the density μ belongs to the dual Sobolev space $(W_0^{1,2}(\Omega))'$.

4.1 Sobolev spaces

Before starting with the variational problem, we explain the setting where the energy functional is minimized: the Sobolev space $W_0^{1,2}(\Omega)$. More generally, we consider Sobolev spaces associated to any exponent $1 \leqslant q < +\infty$ as follows:

Definition 4.1. Let Ω be a bounded open set, and let $u \in L^q(\Omega)$. We say that u belongs to the *Sobolev space* $W_0^{1,q}(\Omega)$ if there exists $G \in L^q(\Omega; \mathbb{R}^N)$ such that, for every $\Phi \in C^\infty(\overline{\Omega}; \mathbb{R}^N)$, we have

$$\int_\Omega u \operatorname{div} \Phi = \int_\Omega G \cdot \Phi.$$

[1] "The very special conditions in which Hilbert studies the Dirichlet problem seem to be an essential part of the argument, and leave little hope that the general problem could be handled without substantial changes."

Both integrands are summable over Ω since the domain is assumed to be bounded. Two functions $G_1, G_2 \in L^q(\Omega; \mathbb{R}^N)$ satisfying this identity for the same function u are equal almost everywhere in Ω, see Corollary 4.24 in [53] or Theorem 4.3.10 in [345]: this important property is sometimes called the *fundamental theorem of the calculus of variations*. We systematically use the notation

$$\boxed{\nabla u = -G}$$

for the *weak gradient*. We then recover the formula

$$\int_\Omega u \operatorname{div} \Phi = -\int_\Omega \nabla u \cdot \Phi$$

which follows from the divergence theorem for functions $u \in C_0^\infty(\overline{\Omega})$ vanishing on the boundary of a smooth bounded open set Ω.

The Sobolev space $W_0^{1,q}(\Omega)$ is well suited to study minimization problems and to give a meaning to weak formulations of Dirichlet problems involving a zero boundary condition. The reason is that it enjoys two fundamental properties: it is complete with respect to its natural norm satisfying

$$\|u\|_{W^{1,q}(\Omega)}^q = \|u\|_{L^q(\Omega)}^q + \|\nabla u\|_{L^q(\Omega)}^q, \tag{4.1}$$

and is sensitive to boundary conditions.

Exercise 4.1 (zero boundary datum). Let Ω be a smooth bounded open set, and let $u \in C^1(\overline{\Omega})$. Prove that $u \in W_0^{1,q}(\Omega)$ if and only if $u = 0$ on $\partial\Omega$.

The completeness of the Sobolev spaces relies on a stability property satisfied by sequences of Sobolev functions.

Proposition 4.2. *For every bounded open set Ω, $W_0^{1,q}(\Omega)$ is a complete metric space.*

Proof. Given a Cauchy sequence $(u_n)_{n\in\mathbb{N}}$ in $W_0^{1,q}(\Omega)$, the sequence $(u_n)_{n\in\mathbb{N}}$ is Cauchy in $L^q(\Omega)$ and the sequence $(\nabla u_n)_{n\in\mathbb{N}}$ is Cauchy in $L^q(\Omega; \mathbb{R}^N)$, and so they converge to $u \in L^q(\Omega)$ and to $F \in L^q(\Omega; \mathbb{R}^N)$, respectively. To conclude the proof, we show that

$$u \in W_0^{1,q}(\Omega) \quad \text{and} \quad \nabla u = F \tag{4.2}$$

using the *stability property*: for every $\Phi \in C^\infty(\overline{\Omega}; \mathbb{R}^N)$ and every $n \in \mathbb{N}$, we have

$$\int_\Omega u_n \operatorname{div} \Phi = -\int_\Omega \nabla u_n \cdot \Phi,$$

whence, as $n \to \infty$, we get

$$\int_\Omega u \operatorname{div} \Phi = -\int_\Omega F \cdot \Phi.$$

Thus, assertion (4.2) holds, and the conclusion follows. □

The stability property provides one with examples of Sobolev functions which need not be differentiable:

Exercise 4.2. Prove that the function $u\colon B(0;1) \to \mathbb{R}$ defined for $x \in B(0;1)$ by

$$u(x) = 1 - |x|$$

belongs to $W_0^{1,q}(B(0;1))$ for every exponent $1 \leqslant q < +\infty$, and $\nabla u(x) = -x/|x|$ almost everywhere in $B(0;1)$.

Exercise 4.3. Let $\alpha > 0$. Prove that the function $u\colon B(0;1) \to \mathbb{R}$ defined for $x \neq 0$ by

$$u(x) = \frac{1}{|x|^\alpha} - 1$$

belongs to $W_0^{1,q}(B(0;1))$ for every exponent q such that $q(\alpha + 1) < N$.

The convolution product gives a convenient way to study properties of Sobolev functions in $W_0^{1,q}(\Omega)$ via approximation by smooth functions with compact support in $\mathbb{R}^N$:

Proposition 4.3. *Let Ω be a bounded open set. If $u \in W_0^{1,q}(\Omega)$, then, for every $\rho \in C_c^\infty(\mathbb{R}^N)$, we have $\rho * u \in C_c^\infty(\mathbb{R}^N)$ and*

$$\nabla(\rho * u) = \rho * \nabla u \quad \text{in } \mathbb{R}^N.$$

Proof. For every $x \in \mathbb{R}^N$, we have

$$\rho * u(x) = \int_\Omega \rho(x-y)u(y)\,\mathrm{d}y.$$

Thus, $\rho * u \in C_c^\infty(\mathbb{R}^N)$, and by differentiation under the integral sign we get

$$\nabla(\rho * u)(x) = \int_\Omega \nabla_x \rho(x-y)u(y)\,\mathrm{d}y = -\int_\Omega \nabla_y \rho(x-y)u(y)\,\mathrm{d}y.$$

For every $e \in \mathbb{R}^N$, we have $e \cdot \nabla\rho = \operatorname{div}(\rho e)$. Thus, by the linearity of the integral and by the definition of the weak gradient of u we get

$$\begin{aligned} e \cdot \nabla(\rho * u)(x) &= -\int_\Omega \operatorname{div}_y(\rho e)(x-y)\,u(y)\,\mathrm{d}y \\ &= \int_\Omega \rho(x-y)\,e \cdot \nabla u(y)\,\mathrm{d}y \\ &= e \cdot (\rho * \nabla u)(x). \end{aligned}$$

Since this identity holds for every $e \in \mathbb{R}^N$, we have the conclusion. □

Exercise 4.4 (constant Sobolev functions). Prove that if $u \in W_0^{1,q}(\Omega)$ is such that $\nabla u = 0$, then $u = 0$ almost everywhere in Ω.

Exercise 4.5 (Leibniz rule). Prove that, for every $u, v \in W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$, we have $uv \in W_0^{1,q}(\Omega)$ and

$$\nabla(uv) = \nabla u\, v + u\, \nabla v.$$

Exercise 4.6 (chain rule). Let $H \in C^1(\mathbb{R})$. Prove that if $H(0) = 0$ and H' is bounded, then, for every $u \in W_0^{1,q}(\Omega)$, we have $H(u) \in W_0^{1,q}(\Omega)$ and

$$\nabla H(u) = H'(u)\nabla u.$$

In smooth domains Ω, the approximation of Sobolev functions can be performed using functions compactly supported in Ω:

Proposition 4.4. *For every smooth bounded open set Ω, the set $C_c^\infty(\Omega)$ is dense in $W_0^{1,q}(\Omega)$.*

We recover in this case the usual approach to define the space $W_0^{1,q}(\Omega)$ as the completion of $C_c^\infty(\Omega)$ with respect to the Sobolev norm (4.1). We take this statement for granted. Proposition 4.4 can be proved combining the characterization of the kernel of the trace operator (see Theorem 6.6.4 in [187]) with Exercise 15.1 below. Note however that if Ω is a ball, then Proposition 4.4 has a straighforward proof based on scaling:

Exercise 4.7 (approximation with compact support). For every $u \in W_0^{1,q}(B(0;1))$, prove that there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in the set $C_c^\infty(B(0;1))$ which converges to u in $W_0^{1,q}(B(0;1))$.

The fact that nonzero constant functions cannot belong to $W_0^{1,q}(\Omega)$ is quantified by the Poincaré inequality, see Corollary 9.19 in [53] or Theorem 6.4.7 in [345]:

Proposition 4.5. *Let Ω be a bounded open set. Then, for every $u \in W_0^{1,q}(\Omega)$, we have*

$$\|u\|_{L^q(\Omega)} \leqslant C\|\nabla u\|_{L^q(\Omega)},$$

for some constant $C > 0$ depending on the diameter $\operatorname{diam}\Omega$.

Proof. Let $Q(a;r)$ be a cube such that $\Omega \Subset Q(a;r)$. Using the fundamental theorem of calculus and the Hölder inequality, one shows that, for every $\varphi \in C_c^\infty(Q(a;r))$,

$$\|\varphi\|_{L^q(Q(a;r))} \leqslant r\|\nabla\varphi\|_{L^q(Q(a;r))}.$$

Given a sequence of mollifiers $(\rho_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^N)$ such that $\Omega+\operatorname{supp}\rho_n \Subset Q(a;r)$, we have $\rho_n * u \in C_c^\infty(Q(a;r))$. By the inequality above for smooth functions and by Proposition 4.3, we then have

$$\|\rho_n * u\|_{L^q(Q(a;r))} \leqslant r\|\rho_n * \nabla u\|_{L^q(Q(a;r))}.$$

Letting $n \to \infty$, the conclusion follows. □

Taking advantage of the Hilbert space structure of $W_0^{1,2}(\Omega)$, we now establish the existence of solutions of the linear Dirichlet problem for the Poisson equation for every density μ in the dual space $(W_0^{1,2}(\Omega))'$:

$$\begin{cases} -\Delta u = \mu & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

The main ingredient is the Fréchet–Riesz representation theorem in Hilbert spaces, see Theorem 5.5 in [53] or Theorem 5.3.1 in [345].

Proposition 4.6. *Let Ω be a bounded open set. Then, for every $\mu \in (W_0^{1,2}(\Omega))'$, there exists a unique function $u \in W_0^{1,2}(\Omega)$ such that, for every $z \in W_0^{1,2}(\Omega)$, we have*

$$\int_\Omega \nabla u \cdot \nabla z = \mu[z].$$

Proof. By the Poincaré inequality (Proposition 4.5), the bilinear form

$$W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \ni (u,z) \longmapsto \int_\Omega \nabla u \cdot \nabla z$$

is an inner product in $W_0^{1,2}(\Omega)$ and induces a norm that is equivalent to the $W^{1,2}$ norm. By the Fréchet–Riesz representation theorem in Hilbert spaces, there exists a unique function $u \in W_0^{1,2}(\Omega)$ such that, for every $z \in W_0^{1,2}(\Omega)$, we have

$$\mu[z] = \int_\Omega \nabla u \cdot \nabla z. \qquad \square$$

A function $u \in W_0^{1,2}(\Omega)$ satisfying the conclusion of Proposition 4.6 is called a (*variational*) *solution* of the Dirichlet problem in $W_0^{1,2}(\Omega)$ with density μ. If u and μ are smooth functions in Ω, then u satisfies pointwise the Poisson equation with density μ. Assuming in addition that u has a smooth extension to the boundary of a smooth domain Ω, then from the integral formulation above we find that such an extension vanishes identically on $\partial\Omega$ (cf. Exercise 4.1), whence u satisfies the Dirichlet problem in the classical sense.

This formalism includes the case of L^2 data:

Example 4.7. Every function $\mu \in L^2(\Omega)$ can be interpreted as an element of the dual space $(W_0^{1,2}(\Omega))'$ by acting on every $z \in W_0^{1,2}(\Omega)$ as

$$\mu[z] = \int_\Omega \mu z.$$

Indeed, by the Hölder inequality we have $\mu z \in L^1(\Omega)$ and

$$|\mu[z]| = \left|\int_\Omega \mu z\right| \leqslant \|\mu\|_{L^2(\Omega)} \|z\|_{L^2(\Omega)} \leqslant \|\mu\|_{L^2(\Omega)} \|z\|_{W^{1,2}(\Omega)}.$$

Thus, the solution of the Dirichlet problem in $W_0^{1,2}(\Omega)$ with density μ exists and satisfies, for every $\zeta \in C_0^\infty(\overline{\Omega})$,

$$-\int_\Omega u\, \Delta\zeta = \int_\Omega \nabla u \cdot \nabla\zeta = \int_\Omega \mu\zeta.$$

Exercise 4.8 (weak maximum principle). Prove that if $\mu \in L^2(\Omega)$ is a nonnegative function, then the solution of the Dirichlet problem in $W_0^{1,2}(\Omega)$ with density μ is also nonnegative almost everywhere.

The Fréchet–Riesz representation theorem is well suited to solve *linear* problems. In the study of *nonlinear* problems, the Rellich–Kondrashov compactness theorem is an important tool to deal with bounded sequences of Sobolev functions, see Theorem 9.16 in [53], Theorem 5.7.1 in [125], or Theorem 6.4.6 in [345]:

Proposition 4.8. *Let Ω be a bounded open set. Then, for every bounded sequence $(u_n)_{n\in\mathbb{N}}$ in $W_0^{1,q}(\Omega)$, there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ converging strongly in $L^q(\Omega)$.*

The proof relies on M. Riesz's compactness criterion of equi-integrable sequences of functions, see Theorem 4.26 in [53] or Theorem 4.4.2 in [345], which is the counterpart in Lebesgue spaces of the classical Ascoli–Arzelà compactness theorem for continuous functions.

The subsequence given by Proposition 4.8 can be assumed to converge almost everywhere. To be more precise, we may extract a further subsequence from $(u_{n_k})_{k\in\mathbb{N}}$ which converges almost everywhere in Ω. This is a general property of sequences of functions converging strongly in Lebesgue spaces, and relies on the following partial converse of the dominated convergence theorem, see Theorem 4.9 in [53] or Proposition 4.2.10 in [345]:

Chapter 5
Linear regularity theory

> "Si la force F est proportionnelle à $\frac{1}{\Delta^2}$, il suffira de trouver la valeur de $\frac{1}{\Delta}$ et de la différencier par les méthodes ordinaires."[1]
>
> Joseph-Louis Lagrange

We investigate the Sobolev regularity of solutions of the linear Dirichlet problem when the density is merely an L^1 function and, more generally, a finite Borel measure.

5.1 Embedding in Sobolev spaces

We follow Littman, Stampacchia and Weinberger's duality approach to prove the Sobolev regularity of solutions of the linear Dirichlet problem

$$\begin{cases} -\Delta u = \mu & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

involving measure data, see Theorem 5.1 in [213] or Théorème 9.1 in [316]:

Proposition 5.1. *Let Ω be a smooth bounded open set, and let $\mu \in \mathcal{M}(\Omega)$. If u is the solution of the linear Dirichlet problem with density μ, then, for every $1 \leqslant q < \frac{N}{N-1}$, we have $u \in W_0^{1,q}(\Omega)$, and the estimate*

$$\|u\|_{W^{1,q}(\Omega)} \leqslant C\|\mu\|_{\mathcal{M}(\Omega)}$$

holds for some constant $C > 0$ depending on q, N, and Ω.

By the Sobolev embedding (see Corollary 4.12), the solution u thus belongs to the Lebesgue space $L^p(\Omega)$ for every exponent $1 \leqslant p < \frac{N}{N-2}$, and satisfies the estimate

$$\|u\|_{L^p(\Omega)} \leqslant C'\|\mu\|_{\mathcal{M}(\Omega)}.$$

[1] "Assuming the force F is proportional to $\frac{1}{\Delta^2}$, it suffices to find the value of $\frac{1}{\Delta}$ and to differentiate it by ordinary methods."

On the other hand, solutions need not belong to $W_0^{1,\frac{N}{N-1}}(\Omega)$, and this is related to the failure of the natural counterpart of the Calderón–Zygmund regularity theory for L^1 data that we explain at the end of this section.

The introduction of the potential function u by Lagrange [189] – later pursued by Laplace [190], [191] and Poisson [283], [28] – was originally motivated by the study of the force field $G = -\nabla u$. In this respect, the scalar quantity u is supposedly easier to compute, but the force G itself is the major physical notion. The embedding of solutions of the Dirichlet problem into Sobolev spaces thus ensures the existence of the force field G, with an estimate in terms of the total mass or the total electric charge $\|\mu\|_{\mathcal{M}(\Omega)} = |\mu|(\Omega)$.

Proposition 5.1 relies on the following estimate due to Stampacchia (Proposizione 5.1 in [315]), in the spirit of the celebrated works of De Giorgi [103] and Nash [260] providing Hölder continuity of solutions of elliptic PDEs. The strategy of the proof below by Hartman and Stampacchia (Lemma 7.3 in [161]) is based on Stampacchia's truncation method.

Lemma 5.2. *Let Ω be a bounded open set. If $v \in W_0^{1,2}(\Omega)$ satisfies the linear Dirichlet problem*

$$\begin{cases} -\Delta v = f + \operatorname{div} F & \text{on } \Omega, \\ \quad v = 0 & \text{on } \partial\Omega, \end{cases}$$

for some $f \in L^r(\Omega;\mathbb{R})$ and some $F \in L^r(\Omega;\mathbb{R}^N)$ with $r > N$, then $v \in L^\infty(\Omega)$, and the estimate

$$\|v\|_{L^\infty(\Omega)} \leqslant C(\|f\|_{L^r(\Omega)} + \|F\|_{L^r(\Omega)}),$$

holds for some constant $C > 0$ depending on r, N and Ω.

Proof of Lemma 5.2. We assume in the proof that $N \geqslant 3$; the case of dimension $N = 2$ requires some small modification concerning the Sobolev inequality. Given $\kappa > 0$, let $S_\kappa\colon \mathbb{R} \to \mathbb{R}$ be the function defined for $t \in \mathbb{R}$ by

$$S_\kappa(t) = \begin{cases} t + \kappa & \text{if } t < -\kappa, \\ 0 & \text{if } -\kappa \leqslant t \leqslant \kappa, \\ t - \kappa & \text{if } t > \kappa. \end{cases}$$

Note that $S_\kappa(v) \in W_0^{1,2}(\Omega)$ and (cf. Exercise 5.3)

$$|\nabla S_\kappa(v)|^2 = \nabla v \cdot \nabla S_\kappa(v).$$

Taking $S_\kappa(v)$ as a test function of the Dirichlet problem, we get

$$\int_\Omega |\nabla S_\kappa(v)|^2 = \int_\Omega f S_\kappa(v) - \int_\Omega F \cdot \nabla S_\kappa(v).$$

Since $S_\kappa(v) = 0$ in $\{|v| \leqslant \kappa\}$, the Hölder inequality yields

$$\int_\Omega |\nabla S_\kappa(v)|^2 \leqslant (\|f\|_{L^2(\{|v|>\kappa\})} + \|F\|_{L^2(\{|v|>\kappa\})})\|S_\kappa(v)\|_{W^{1,2}(\Omega)}.$$

Using the Poincaré inequality (Proposition 4.5), we deduce that

$$\|\nabla S_\kappa(v)\|_{L^2(\Omega)} \leqslant C_1(\|f\|_{L^2(\{|v|>\kappa\})} + \|F\|_{L^2(\{|v|>\kappa\})}).$$

On the other hand, by the Hölder and the Sobolev inequalities (Corollary 4.12), we have

$$\begin{aligned}\|S_\kappa(v)\|_{L^1(\Omega)} &\leqslant \|S_\kappa(v)\|_{L^{\frac{2N}{N-2}}(\Omega)} \,|\{|v| > \kappa\}|^{\frac{1}{2}+\frac{1}{N}}\\ &\leqslant C_2\|\nabla S_\kappa(v)\|_{L^2(\Omega)} \,|\{|v| > \kappa\}|^{\frac{1}{2}+\frac{1}{N}}.\end{aligned}$$

Combining the two estimates we deduce that, for every $\kappa > 0$,

$$\|S_\kappa(v)\|_{L^1(\Omega)} \leqslant C_3(\|f\|_{L^2(\{|v|>\kappa\})} + \|F\|_{L^2(\{|v|>\kappa\})}) \,|\{|v| > \kappa\}|^{\frac{1}{2}+\frac{1}{N}}.$$

By the Hölder inequality, for every $r \geqslant 2$ we have

$$\|f\|_{L^2(\{|v|>\kappa\})}+\|F\|_{L^2(\{|v|>\kappa\})} \leqslant (\|f\|_{L^r(\{|v|>\kappa\})}+\|F\|_{L^r(\{|v|>\kappa\})}) \,|\{|v| > \kappa\}|^{\frac{1}{2}-\frac{1}{r}}.$$

Therefore,

$$\|S_\kappa(v)\|_{L^1(\Omega)} \leqslant C_3(\|f\|_{L^r(\Omega)} + \|F\|_{L^r(\Omega)}) \,|\{|v| > \kappa\}|^{1+\frac{1}{N}-\frac{1}{r}}.$$

Claim. *If there exist $\alpha > 1$ and $A \geqslant 0$ such that, for every $\kappa > 0$,*

$$\|S_\kappa(v)\|_{L^1(\Omega)} \leqslant A \,|\{|v| > \kappa\}|^\alpha,$$

then $v \in L^\infty(\Omega)$ and

$$\|v\|_{L^\infty(\Omega)} \leqslant C' A^{\frac{1}{\alpha}} \|v\|_{L^1(\Omega)}^{1-\frac{1}{\alpha}}.$$

Assuming the claim, we can conclude the proof of the lemma. Indeed, since $r > N$ and $\|v\|_{L^1(\Omega)} \leqslant |\Omega|\|v\|_{L^\infty(\Omega)}$, we deduce from the claim that

$$\|v\|_{L^\infty(\Omega)} \leqslant C_4(\|f\|_{L^r(\Omega)} + \|F\|_{L^r(\Omega)}),$$

which is the estimate we wanted to establish.

Proof of the claim. By Cavalieri's principle (Proposition 1.7),

$$\|S_\kappa(v)\|_{L^1(\Omega)} = \int_0^\infty |\{|S_\kappa(v)| \geqslant s\}| \, \mathrm{d}s = \int_\kappa^\infty |\{|v| \geqslant s\}| \, \mathrm{d}s.$$

Therefore, we may rewrite the assumption on v as

$$\int_\kappa^\infty |\{|v| \geqslant s\}| \, \mathrm{d}s \leqslant A \, |\{|v| \geqslant \kappa\}|^\alpha .$$

Let $H \colon [0, +\infty) \to \mathbb{R}$ be the function defined for $t \geqslant 0$ by

$$H(t) = \int_t^\infty |\{|v| \geqslant s\}| \, \mathrm{d}s.$$

The function $[0, +\infty) \ni s \mapsto |\{|v| \geqslant s\}|$ is nonincreasing, and whence continuous except for countably many points. Thus, for almost every $t \geqslant 0$, we have

$$H'(t) = -\,|\{|v| \geqslant t\}| \, .$$

In view of the estimate, for almost every $\kappa \geqslant 0$ we then have

$$-H'(\kappa) = |\{|v| \geqslant \kappa\}| \geqslant \left[\frac{H(\kappa)}{A}\right]^{\frac{1}{\alpha}} .$$

Integrating this inequality (Exercise 5.1), we conclude that if $\alpha > 1$, then $H(\kappa_0) = 0$ for some $\kappa_0 \geqslant 0$ such that

$$\kappa_0 \leqslant C_5 A^{\frac{1}{\alpha}} H(0)^{1-\frac{1}{\alpha}} .$$

Since $\|v\|_{L^\infty(\Omega)} \leqslant \kappa_0$ and $H(0) = \|v\|_{L^1(\Omega)}$, the claim follows. △

The proof of the lemma is complete. □

Exercise 5.1 (finite time vanishing). Let $\beta < 1$, and let $H \colon [0, +\infty) \to \mathbb{R}$ be a nonnegative absolutely continuous function such that

$$H' \leqslant -BH^\beta$$

almost everywhere in $[0, +\infty)$, for some constant $B > 0$. Prove that if $s > 0$ is such that $H(s) > 0$, then, for every $0 \leqslant t \leqslant s$, we have

$$H(t) \leqslant (H(0)^{1-\beta} - (1-\beta)Bt)^{\frac{1}{1-\beta}} .$$

Deduce that $H(t) = 0$ for every $t \geqslant \frac{H(0)^{1-\beta}}{(1-\beta)B}$.

The existence of the weak gradient ∇u as an element in the Lebesgue space $L^q(\Omega)$ is obtained using the Riesz representation theorem (Proposition 3.3) and Stampacchia's estimate (Lemma 5.2).

Proof of Proposition 5.1. For every $\zeta \in C_0^\infty(\overline{\Omega})$, the solution of the linear Dirichlet problem with density μ satisfies the inequality

$$\left| \int_\Omega u \, \Delta\zeta \right| \leqslant \|\mu\|_{\mathcal{M}(\Omega)} \|\zeta\|_{L^\infty(\Omega)}.$$

Given $f \in C^\infty(\overline{\Omega})$, by the assumption on Ω we may use as test function $\zeta \in C_0^\infty(\overline{\Omega})$ the solution of the linear Dirichlet problem with density f. For every $1 < q < \frac{N}{N-1}$, the conjugate exponent satisfies $q' > N$, whence by Stampacchia's estimate we have

$$\|\zeta\|_{L^\infty(\Omega)} \leqslant C \|f\|_{L^{q'}(\Omega)}.$$

Therefore,

$$\left| \int_\Omega uf \right| \leqslant C \|\mu\|_{\mathcal{M}(\Omega)} \|f\|_{L^{q'}(\Omega)}.$$

This estimate holds for every $f \in C^\infty(\overline{\Omega})$, and then, by weak density, for every $f \in L^\infty(\Omega)$. We deduce from the Riesz representation theorem (Exercise 3.1) that $u \in L^q(\Omega)$ and

$$\|u\|_{L^q(\Omega)} \leqslant C \|\mu\|_{\mathcal{M}(\Omega)}.$$

We now prove that u has a weak derivative in $L^q(\Omega; \mathbb{R}^N)$. For this purpose, given $F \in C^\infty(\overline{\Omega}; \mathbb{R}^N)$, we use as test function $\zeta \in C_0^\infty(\overline{\Omega})$ the solution of the linear Dirichlet problem with density div F. For every $1 < q < \frac{N}{N-1}$, it follows from Stampacchia's estimate that

$$\left| \int_\Omega u \operatorname{div} F \right| \leqslant \|\mu\|_{\mathcal{M}(\Omega)} \|\zeta\|_{L^\infty(\Omega)} \leqslant C \|\mu\|_{\mathcal{M}(\Omega)} \|F\|_{L^{q'}(\Omega)}.$$

Thus, the functional

$$C^\infty(\overline{\Omega}; \mathbb{R}^N) \ni F \longmapsto \int_\Omega u \operatorname{div} F$$

admits a unique continuous linear extension in $L^{q'}(\Omega; \mathbb{R}^N)$. By the Riesz representation theorem (Proposition 3.3), there exists a unique function $G \in L^q(\Omega; \mathbb{R}^N)$ such that, for every $F \in C^\infty(\overline{\Omega}; \mathbb{R}^N)$,

$$\int_\Omega u \operatorname{div} F = \int_\Omega G \cdot F.$$

Therefore, $u \in W_0^{1,q}(\Omega)$ and we have

$$\|G\|_{L^q(\Omega)} \leqslant C\|\mu\|_{\mathcal{M}(\Omega)}.$$

The proof of the proposition is complete when $q > 1$, and by the Hölder inequality we also get the conclusion for $q = 1$. □

In the previous proof, we assumed that a solution u exists, and this follows from Proposition 3.2, which is also based on Stampacchia's estimate. We can in fact prove simultaneously the existence and regularity of solutions of the linear Dirichlet problem using a compactness argument. Indeed, the previous proof gives the estimate we need: for every $1 \leqslant q < \frac{N}{N-1}$ and every $v \in W_0^{1,2}(\Omega)$ satisfying the Dirichlet problem with density $\nu \in L^2(\Omega)$,

$$\|v\|_{W^{1,q}(\Omega)} \leqslant C\|\nu\|_{L^1(\Omega)}.$$

Next, if $(\mu_n)_{n\in\mathbb{N}}$ is a sequence in $L^2(\Omega)$ converging in the sense of measures to μ (Proposition 2.7) and if u_n is the solution of the Dirichlet problem in $W_0^{1,2}(\Omega)$ with density μ_n (Proposition 4.6), then by the above estimate the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $W_0^{1,q}(\Omega)$. By the Rellich–Kondrashov compactness theorem (Proposition 4.8), we may extract a subsequence converging strongly in $L^q(\Omega)$ to some function u which satisfies the Dirichlet problem with density μ. By the closure property of Sobolev spaces (Proposition 4.10), we deduce that $u \in W_0^{1,q}(\Omega)$ for $q > 1$, and so also for $q = 1$. This implies that *every* solution of the linear Dirichlet problem has the required regularity since solutions are unique (Proposition 3.5).

We now compare the Sobolev embedding of solutions of the linear Dirichlet problem

$$\begin{cases} -\Delta u = \mu & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases}$$

in terms of L^1 or measure data μ (Proposition 5.1) with the classical Calderón–Zygmund L^p theory for $1 < p < +\infty$, see Theorem 9.15 and Lemma 9.17 in [146]:

Proposition 5.3. *Let $1 < p < +\infty$, and let Ω be a smooth bounded open set. If $\mu \in L^p(\Omega)$, then the solution u of the Dirichlet problem above belongs to $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, and the estimate*

$$\|u\|_{W^{2,p}(\Omega)} \leqslant C\|\mu\|_{L^p(\Omega)}$$

holds for some constant $C > 0$ depending on p and Ω.

This is a remarkable contribution to the regularity theory of elliptic PDEs, and more generally to harmonic analysis through estimates on the Riesz transform. The inequality above fails for $p = 1$, which is the case handled by Proposition 5.1 above:

Exercise 5.2 (failure of L^1 theory). Let $N \geqslant 3$, and let $\varphi \in C_c^\infty(B(0;1))$ be such that $\varphi = 1$ in some neighborhood of 0. Prove that, for every $0 < \alpha \leqslant 1$, the function $u\colon B(0;1) \to \mathbb{R}$ defined by

$$u(x) = \frac{1}{|x|^{N-2}}\Big(\log\frac{1}{|x|}\Big)^{-\alpha}\varphi,$$

satisfies the Dirichlet problem with L^1 density in $B(0;1)$, but $u \notin W^{2,1}(B(0;1))$.

One can also argue by contradiction as follows. If the estimate

$$\|u\|_{W^{2,1}(\Omega)} \leqslant C\|\mu\|_{L^1(\Omega)}$$

were correct in dimension $N \geqslant 3$ for every solution of the linear Dirichlet problem, then by the Sobolev–Gagliardo–Nirenberg inequality (Corollary 4.12) we would have

$$\|u\|_{L^{\frac{N}{N-2}}(\Omega)} \leqslant C'\|\mu\|_{L^1(\Omega)}.$$

By an approximation argument (Proposition 2.7), this inequality would also hold with μ replaced by a Dirac mass δ_a, in which case $\|\mu\|_{L^1(\Omega)}$ should be replaced by the total mass $\|\delta_a\|_{\mathcal{M}(\Omega)} = 1$. In particular, we would have $u \in L^{\frac{N}{N-2}}(\Omega)$. But this is not possible since, in the unit ball $B(0;1)$, the solution of the linear Dirichlet problem with density δ_0 is explicitly given by

$$u(x) = \frac{1}{(N-2)\sigma_N}\Big(\frac{1}{\|x\|^{N-2}} - 1\Big),$$

which does not belong to $L^{\frac{N}{N-2}}(B(0;1))$. In fact, a deep construction of Ornstein implies that even the stronger inequality

$$\|D^2u\|_{L^1(\Omega)} \leqslant \sum_{i=1}^{N}\Big\|\frac{\partial^2 u}{\partial x_i^2}\Big\|_{L^1(\Omega)}$$

is false, see Theorem 1.3 in [182] or Theorem 1 in [268].

The connection with the Calderón–Zygmund's singular integral theory becomes more transparent by considering the following weak type estimates that we establish later on in this chapter (Proposition 5.7). Firstly, we have a weak $L^{\frac{N}{N-2}}$ estimate for solutions of the Dirichlet problem: for every $t > 0$,

$$t\,|\{|u| \geqslant t\}|^{\frac{N-2}{N}} \leqslant C\|\Delta u\|_{L^1(\Omega)}$$

and, secondly, a weak $L^{\frac{N}{N-1}}$ estimate for the first-order derivatives of solutions: for every $t > 0$,

$$t\,|\{|\nabla u| \geqslant t\}|^{\frac{N-1}{N}} \leqslant C'\|\Delta u\|_{L^1(\Omega)}.$$

These estimates are the natural companions of the following weak L^1 estimate for second derivatives of solutions: for every $t > 0$,

$$t\,\big|\{|D^2 u| \geqslant t\}\big| \leqslant C''\|\Delta u\|_{L^1(\Omega)},$$

that lies at the heart of Calderón and Zygmund's approach, see eq. (9.30) in [146] or p. 30, eq. (9), in [317]. Once the latter weak estimate is established, the classical L^p estimates for $1 < p < +\infty$ then follow by interpolation of continuous linear operators in Lebesgue spaces.

5.2 Weak Lebesgue functions

Before establishing the weak estimates satisfied by solutions of the linear Dirichlet problem and their first-order derivatives, we recall the definition and some basic properties of weak Lebesgue functions.

Definition 5.4. Let $1 \leqslant p < +\infty$. A Borel measurable function $u\colon \Omega \to \mathbb{R}$ is a *weak L^p function* if there exists $M \geqslant 0$ such that, for every $t > 0$,

$$t\,|\{|u| \geqslant t\}|^{\frac{1}{p}} \leqslant M.$$

For example, if $u \in L^p(\Omega)$, then, for every $t > 0$, by the Chebyshev inequality we have

$$t\,|\{|u| \geqslant t\}|^{\frac{1}{p}} \leqslant \|u\|_{L^p(\Omega)},$$

whence u is a weak L^p function, but the converse is false as one can see by considering the function

$$B(0;1) \ni x \longmapsto \frac{1}{|x|^{\frac{N}{p}}}.$$

We are actually just missing the embedding into the Lebesgue space $L^p(\Omega)$:

Proposition 5.5. *If $u\colon \Omega \to \mathbb{R}$ is a weak L^p function and if Ω has finite Lebesgue measure, then, for every $1 \leqslant r < p$, we have $u \in L^r(\Omega)$ and*

$$\|u\|_{L^r(\Omega)} \leqslant CM,$$

for some constant $C > 0$ depending on p, r and $|\Omega|$.

Proof. By Cavalieri's principle (1.5),

$$\int_\Omega |u|^r = r\int_0^\infty t^{r-1}\,|\{|u| \geqslant t\}|\,\mathrm{d}t.$$

We estimate the measure $|\{|u| \geqslant t\}|$ for t small by $|\Omega|$ and for t large by M^p/t^p. Since $r < p$, for every $s > 0$ we have

$$\int_\Omega |u|^r \leqslant r\int_0^s t^{r-1}\,|\Omega|\,\mathrm{d}t + r\int_s^\infty t^{r-1}\frac{M^p}{t^p}\,\mathrm{d}t = s^r\,|\Omega| + \frac{rM^p}{p-r}\frac{1}{s^{p-r}}.$$

Thus, $u \in L^r(\Omega)$. The inequality is obtained by minimizing the right-hand side with respect to s. □

We now give a convenient characterization of weak L^p functions in the spirit of the Hölder inequality. The proof requires the same trick as in Proposition 5.5, and is used again in the proof of the weak elliptic estimates in Proposition 5.7.

Proposition 5.6. *Let $u\colon \Omega \to \mathbb{R}$ be a Borel measurable function and $1 < p < +\infty$. We have that u is a weak L^p function if and only if, for every Borel set $A \subset \Omega$,*

$$\int_A |u| \leqslant M'\,|A|^{\frac{p-1}{p}},$$

for some constant $M' \geqslant 0$.

Proof. If u satisfies the estimate, then taking $A = \{|u| \geqslant t\}$ with $t > 0$ we have

$$\int_{\{|u|\geqslant t\}} |u| \leqslant M'\,|\{|u| \geqslant t\}|^{\frac{p-1}{p}}.$$

Thus, by the Chebyshev inequality,

$$t\,|\{|u| \geqslant t\}| \leqslant M'\,|\{|u| \geqslant t\}|^{\frac{p-1}{p}}.$$

Assuming that the sets $\{|u| \geqslant t\}$ have finite measure, we deduce that u is a weak L^p function. We can avoid such a finiteness assumption by taking instead

$$A_n = \{|u| \geqslant t\} \cap B(0; r_n),$$

where $(r_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers diverging to infinity. For every $n \in \mathbb{N}$, the previous argument gives

$$t\,|A_n|^{\frac{1}{p}} \leqslant M'.$$

As $n \to \infty$, we deduce that $\{|u| \geqslant t\}$ has finite measure, and satisfies the weak L^p estimate.

Conversely, if u is a weak L^p function, then by Cavalieri's principle (Proposition 1.7) we have

$$\int_A |u| = \int_0^\infty |A \cap \{|u| \geqslant t\}| \,\mathrm{d}t.$$

We estimate the measure of the set $A \cap \{|u| \geqslant t\}$ by $|A|$ or $|\{|u| \geqslant t\}|$ according to whether t is small or large: for every $s > 0$,

$$\int_A |u| \leqslant \int_0^s |A| \,\mathrm{d}t + \int_s^\infty |\{|u| \geqslant t\}| \,\mathrm{d}t = s|A| + \frac{M^p}{p-1}\frac{1}{s^{p-1}}.$$

Minimizing the right-hand side with respect to s, we have the estimate. □

From the previous proof, the smallest constants M and M' arising in the definition of weak L^p functions and in the previous proposition are equivalent:

$$cM' \leqslant M \leqslant M',$$

for some constant $c > 0$ depending on p.

We can compare the integral condition that appears in Proposition 5.6 with the Riesz representation theorem (Exercise 3.1). Indeed, assume that we are given a nonnegative summable function $u \colon \Omega \to \mathbb{R}$, and we wish to prove that $u \in L^p(\Omega)$. It then suffices to prove that, for every $v \in L^\infty(\Omega)$, we have

$$\int_\Omega uv \leqslant M'' \|v\|_{L^{\frac{p}{p-1}}(\Omega)}.$$

In Proposition 5.6, only characteristic functions $v = \chi_A$ over Borel sets are admissible, and in this case u is merely a weak L^p function. If we further restrict the class of Borel sets we are allowed to take, for instance to open balls $B(x;r) \subset \Omega$, we fall into the class of Morrey functions satisfying

$$\int_{B(x;r)} u \leqslant M''' r^{N\frac{p-1}{p}}.$$

The latter property has a natural counterpart in the setting of measures, see e.g. Section 2 in [274]. More precisely, given $1 \leqslant p < +\infty$, we say that a locally finite Borel measure μ in $\mathbb{R}^N$ belongs to the Morrey space $\mathcal{M}^p(\mathbb{R}^N)$ if there exists $C > 0$ such that

$$|\mu|(B(x;r)) \leqslant C r^{N\frac{p-1}{p}},$$

for every $x \in \mathbb{R}^N$ and every $r > 0$. We then define the Morrey norm

$$\|\mu\|_{\mathcal{M}^p(\mathbb{R}^N)} = \sup_{\substack{x \in \mathbb{R}^N \\ r > 0}} \frac{|\mu|(B(x;r))}{r^{N\frac{p-1}{p}}}. \tag{5.1}$$

We have for example that $|\mu| \leqslant \alpha \mathcal{H}^s_\infty$ for some $0 \leqslant s < N$ and $\alpha > 0$ if and only if $\mu \in \mathcal{M}^{\frac{N}{N-s}}(\mathbb{R}^N)$ (Proposition B.3), where $\mathcal{H}^s_\infty$ denotes the Hausdorff content of dimension s. Embeddings in Morrey spaces, and their connection with the trace inequality, are investigated in Chapters 10, 16, and 17.

5.3 Critical estimates

We now present an improvement of the Sobolev embedding of solutions of the linear Dirichlet problem in terms of weak Lebesgue estimates. Although the inequalities

$$\|u\|_{L^{\frac{N}{N-2}}(\Omega)} \leqslant C\|\mu\|_{L^1(\Omega)} \quad \text{and} \quad \|\nabla u\|_{L^{\frac{N}{N-1}}(\Omega)} \leqslant C'\|\mu\|_{L^1(\Omega)}$$

are not satisfied, they have a true counterpart in the setting of weak Lebesgue spaces. Such weak estimates are implicitly stated in works by Zygmund (p. 247 in [349]) and Stampacchia (Lemma 7.3 in [315]) using an argument based on the Newtonian potential generated by the measure μ; see also Appendix in [25]. We present a different strategy developed by Boccardo and Gallouët [30] (see also [23]), and based on Stampacchia's truncation argument.

Proposition 5.7. *Let $N \geqslant 3$, let Ω be a smooth bounded open set, and let $\mu \in \mathcal{M}(\Omega)$. If u is the solution of the linear Dirichlet problem with density μ, then, for every $t > 0$, we have*

$$t\,|\{|u| \geqslant t\}|^{\frac{N-2}{N}} \leqslant C\|\mu\|_{\mathcal{M}(\Omega)}$$

and

$$t\,|\{|\nabla u| \geqslant t\}|^{\frac{N-1}{N}} \leqslant C'\|\mu\|_{\mathcal{M}(\Omega)},$$

for some constants $C, C' > 0$ depending on the dimension N.

The constants involved in these critical estimates do not depend on the size of the domain. We rely on Stampacchia's method based on the truncation function

$$T_\kappa : \mathbb{R} \longrightarrow \mathbb{R}$$

defined for $s \in \mathbb{R}$ by

$$T_\kappa(s) = \begin{cases} -\kappa & \text{if } s < -\kappa, \\ s & \text{if } -\kappa \leqslant s \leqslant \kappa, \\ \kappa & \text{if } s > \kappa. \end{cases}$$

Lemma 5.8. *Let Ω be a smooth bounded open set, and let $\mu \in \mathcal{M}(\Omega)$. If u is the solution of the linear Dirichlet problem with density μ, then, for every $\kappa > 0$, we have $T_\kappa(u) \in W_0^{1,2}(\Omega)$ and*

$$\|\nabla T_\kappa(u)\|_{L^2(\Omega)} \leqslant \kappa^{\frac{1}{2}} \|\mu\|_{\mathcal{M}(\Omega)}^{\frac{1}{2}}.$$

Proof of Lemma 5.8. We first assume that $u \in W_0^{1,2}(\Omega)$ and $\mu \in L^2(\Omega)$. By Exercise 5.3 below, for every $\kappa > 0$ we have $T_\kappa(u) \in W_0^{1,2}(\Omega)$ and

$$|\nabla T_\kappa(u)|^2 = \nabla T_\kappa(u) \cdot \nabla u.$$

Using $T_\kappa(u)$ as a test function in the equation satisfied by u we get

$$\int_\Omega |\nabla T_\kappa(u)|^2 = \int_\Omega \nabla T_\kappa(u) \cdot \nabla u = -\int_\Omega T_\kappa(u)\mu \leqslant \kappa \|\mu\|_{L^1(\Omega)}.$$

This gives the estimate when $u \in W_0^{1,2}(\Omega)$ and $\mu \in L^2(\Omega)$.

Given $\mu \in \mathcal{M}(\Omega)$, we consider a sequence of functions $(\mu_n)_{n\in\mathbb{N}}$ in $L^2(\Omega)$ converging weakly to μ in the sense of measures, and such that (Proposition 2.7)

$$\lim_{n\to\infty} \|\mu_n\|_{L^1(\Omega)} = \|\mu\|_{\mathcal{M}(\Omega)}.$$

If $u_n \in W_0^{1,2}(\Omega)$ denotes the solution of the linear Dirichlet problem with density μ_n, then, for every $n \in \mathbb{N}$, we have

$$\|\nabla T_\kappa(u_n)\|_{L^2(\Omega)} \leqslant \kappa^{\frac{1}{2}} \|\mu_n\|_{\mathcal{M}(\Omega)}^{\frac{1}{2}}.$$

Since the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $W_0^{1,1}(\Omega)$ (Proposition 5.1), it follows from the Rellich–Kondrashov compactness theorem (Proposition 4.8) and from the uniqueness of the solution of the Dirichlet problem (Proposition 3.5) that $(u_n)_{n\in\mathbb{N}}$ converges to u in $L^1(\Omega)$. Since $(T_\kappa(u_n))_{n\in\mathbb{N}}$ is bounded in $W_0^{1,2}(\Omega)$, by the closure property in Sobolev spaces (Proposition 4.10) we have that $T_\kappa(u) \in W_0^{1,2}(\Omega)$ and

$$\|\nabla T_\kappa(u)\|_{L^2(\Omega)} \leqslant \liminf_{n\to\infty} \|\nabla T_\kappa(u_n)\|_{L^2(\Omega)}.$$

The conclusion follows. □

Exercise 5.3 (truncation of Sobolev functions). Let $1 \leqslant q < +\infty$. Prove that, for every $u \in W_0^{1,q}(\Omega)$, we have $T_\kappa(u) \in W_0^{1,q}(\Omega)$ and

$$\nabla T_\kappa(u) = \begin{cases} \nabla u & \text{in } \{|u| \leqslant \kappa\}, \\ 0 & \text{in } \{|u| > \kappa\}. \end{cases}$$

Deduce in particular that $|\nabla T_\kappa(u)|^2 = \nabla T_\kappa(u) \cdot \nabla u$.

It follows from the previous lemma that, for every $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ such that $\Delta u \in L^1(\Omega)$, we have

$$\|\nabla u\|_{L^2(\Omega)} \leqslant \|u\|_{L^\infty(\Omega)}^{\frac{1}{2}} \|\Delta u\|_{L^1(\Omega)}^{\frac{1}{2}}. \tag{5.2}$$

This inequality is the borderline case of the Gagliardo–Nirenberg interpolation inequality, see [143] and [265]:

$$\|\nabla u\|_{L^{2q}(\mathbb{R}^N)} \leqslant C \|u\|_{L^\infty(\mathbb{R}^N)}^{\frac{1}{2}} \|D^2 u\|_{L^q(\mathbb{R}^N)}^{\frac{1}{2}}$$

for every $1 \leqslant q < +\infty$ and $u \in W^{2,q}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Proof of Proposition 5.7. We begin with the first estimate. By the interpolation inequality (Lemma 5.8), for every $t > 0$ we have $T_t(u) \in W_0^{1,2}(\Omega)$. Thus, by the Sobolev inequality, we may estimate

$$\|T_t(u)\|_{L^{\frac{2N}{N-2}}(\Omega)} \leqslant C_1 \|\nabla T_t(u)\|_{L^2(\Omega)}.$$

Next, by the Chebyshev inequality,

$$t\, |\{|u| \geqslant t\}|^{\frac{N-2}{2N}} \leqslant \|T_t(u)\|_{L^{\frac{2N}{N-2}}(\Omega)}$$

while, by the interpolation inequality, we have

$$\|\nabla T_t(u)\|_{L^2(\Omega)} \leqslant t^{\frac{1}{2}} \|\mu\|_{\mathcal{M}(\Omega)}^{\frac{1}{2}}.$$

We then deduce that

$$t\, |\{|u| \geqslant t\}|^{\frac{N-2}{2N}} \leqslant C_1 t^{\frac{1}{2}} \|\mu\|_{\mathcal{M}(\Omega)}^{\frac{1}{2}}.$$

We now establish the second estimate. For every $t > 0$ and $s > 0$, we have

$$\{|\nabla u| \geqslant t\} \subset \left\{ \begin{matrix} |\nabla u| \geqslant t \\ |u| \leqslant s \end{matrix} \right\} \cup \{|u| \geqslant s\}.$$

Thus, by the subadditivity of the Lebesgue measure, we get

$$|\{|\nabla u| \geqslant t\}| \leqslant \left| \left\{ \begin{matrix} |\nabla u| \geqslant t \\ |u| \leqslant s \end{matrix} \right\} \right| + |\{|u| \geqslant s\}|\,. \tag{5.3}$$

We already have an estimate of the second term in the right-hand side. In order to deal with the first one, we note that (Exercise 5.3)

$$\left\{ \begin{matrix} |\nabla u| \geqslant t \\ |u| \leqslant s \end{matrix} \right\} = \{|\nabla T_s(u)| \geqslant t\}.$$

By the Chebyshev and the interpolation inequalities (Lemma 5.8), we obtain

$$t\left|\begin{Bmatrix}|\nabla u| \geqslant t\\ |u| \leqslant s\end{Bmatrix}\right|^{\frac{1}{2}} \leqslant \|\nabla T_s(u)\|_{L^2(\Omega)} \leqslant s^{\frac{1}{2}}\|\mu\|_{\mathcal{M}(\Omega)}^{\frac{1}{2}}.$$

From (5.3), we then have

$$|\{|\nabla u| \geqslant t\}| \leqslant \frac{s}{t^2}\|\mu\|_{\mathcal{M}(\Omega)} + \frac{C_2}{s^{\frac{N}{N-2}}}\|\mu\|_{\mathcal{M}(\Omega)}^{\frac{N}{N-2}}.$$

Minimizing the right-hand side with respect to s, we obtain the second estimate. □

The best constant C arising in the first estimate of Proposition 5.7 was computed by Cassani, Ruf, and Tarsi (Theorem 5 in [83]) using Talenti's comparison principle [320]. The counterparts of the estimates of Proposition 5.7 in dimension $N = 2$ are

$$|\{|u| \geqslant t\}| \leqslant C\,|\Omega|\,\mathrm{e}^{-Ct/\|\Delta u\|_{\mathcal{M}(\Omega)}}$$

and

$$|\{|\nabla u| \geqslant t\}|^{\frac{1}{2}} \leqslant \frac{C'}{t}\|\Delta u\|_{\mathcal{M}(\Omega)}.$$

The first one can be obtained as in the previous proof by replacing the Sobolev inequality by the Trudinger inequality [325], see also Theorem 7.15 in [146]:

$$\int_\Omega \mathrm{e}^{\alpha v^2/\|\nabla v\|_{L^2}^2} \leqslant C''\,|\Omega|\,.$$

The second estimate is more subtle. In Lemma A.14 in [25], it relies on the integral representation of ∇u in terms of the Green function G,

$$\nabla u(x) = \frac{1}{2\pi}\int_\Omega \nabla_x G(x, y)\,\mathrm{d}\mu(y),$$

and on the pointwise estimate

$$|\nabla_x G(x, y)| \leqslant \frac{C'''}{|x - y|}.$$

This argument relies on the linearity of the Laplacian and on the integral representation of solutions of the Poisson equation. Recent alternative proofs (Theorem 1.1 in [116], Theorem 1.2 in [178], and Theorem 1.4 in [247]) are based on BMO estimates and reverse Hölder inequalities, but they are not as elementary as in dimension $N \geqslant 3$.

Chapter 6
Comparison tools

"Un ensemble, en chaque point duquel le potentiel atteint sa borne inferieure dans un domaine, ne peut porter de charge positive."[1]

Charles de la Vallée Poussin

We investigate maximum principles adapted to the formalism of weak solutions for the Poisson equation and its companion, the Dirichlet problem.

6.1 Weak maximum principle

We begin with a substitute of the classical weak maximum principle (Corollary 1.10) in the setting of weak solutions:

Proposition 6.1. *Let Ω be a smooth bounded open set and $u \in L^1(\Omega)$. If $-\Delta u \geqslant 0$ in the sense of $(C_0^\infty(\overline{\Omega}))'$, then $u \geqslant 0$ almost everywhere in Ω.*

By $-\Delta u \geqslant 0$ in the sense of $(C_0^\infty(\overline{\Omega}))'$ we mean that, for every nonnegative function $\zeta \in C_0^\infty(\overline{\Omega})$, we have

$$\boxed{-\int_\Omega u\, \Delta\zeta \geqslant 0.}$$

Test functions in $C_0^\infty(\overline{\Omega})$ are sensitive to the information that $u \geqslant 0$ on the boundary $\partial\Omega$:

Example 6.2. Every *nonnegative* superharmonic function $u\colon \mathbb{R}^N \to \mathbb{R}$ satisfies $-\Delta u \geqslant 0$ in the sense of $(C_0^\infty(\overline{\Omega}))'$ in a smooth bounded open set Ω. Indeed, given a sequence of mollifiers $(\rho_k)_{k\in\mathbb{N}}$, each function $\rho_k * u$ is nonnegative and superharmonic in $\mathbb{R}^N$ (Lemma 2.22). In addition, for every nonnegative function $\zeta \in C_0^\infty(\overline{\Omega})$ we have $\frac{\partial\zeta}{\partial n} \leqslant 0$ on $\partial\Omega$. By the divergence theorem, for every $k \in \mathbb{N}$ we thus have

$$\int_\Omega (\rho_k * u)\, \Delta\zeta = \int_\Omega \zeta\, \Delta(\rho_k * u) + \int_{\partial\Omega} (\rho_k * u)\frac{\partial\zeta}{\partial n}\, \mathrm{d}\sigma \leqslant 0.$$

[1] "A set such that in each of its points the potential attains its infimum in a domain cannot carry a positive charge."

The conclusion then follows letting $k \to \infty$. An alternative approach that does not rely on the convolution of u is based on the composition of test functions with convex functions (Lemma 17.6).

In the proof of the proposition, we replace the term $-\Delta\zeta$ by any nonnegative function in $C^\infty(\overline{\Omega})$. This is possible in view of the classical weak maximum principle.

Proof of Proposition 6.1. For every $f \in C^\infty(\overline{\Omega})$, let $\zeta \in C_0^\infty(\overline{\Omega})$ be the solution of the linear Dirichlet problem

$$\begin{cases} -\Delta\zeta = f & \text{in } \Omega, \\ \zeta = 0 & \text{on } \partial\Omega. \end{cases}$$

If $f \geqslant 0$ in $\overline{\Omega}$, then ζ is superharmonic, whence by the classical weak maximum principle we have that $\zeta \geqslant 0$ in Ω. We then deduce that

$$\int_\Omega uf \geqslant 0.$$

Since this inequality holds for every $f \in C^\infty(\overline{\Omega})$ such that $f \geqslant 0$ in $\overline{\Omega}$, we may take a sequence $(f_n)_{n\in\mathbb{N}}$ of such functions converging almost everywhere to the characteristic function $\chi_{\{u<0\}}$, and such that $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$. By the dominated convergence theorem, we deduce that

$$\int_{\{u<0\}} u \geqslant 0.$$

Therefore, $u \geqslant 0$ almost everywhere in Ω. □

It is convenient to pass from an inequality in the sense of distributions to an inequality in the sense of $(C_0^\infty(\overline{\Omega}))'$. Stated differently, we want to find an assumption which ensures that a supersolution of the *equation*

$$-\Delta u = \mu \quad \text{in } \Omega,$$

is a supersolution of the *Dirichlet problem*

$$\begin{cases} -\Delta u = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We first clarify the meaning of the boundary condition, in terms of Sobolev functions, that is implicit in Littman–Stampacchia–Weinberger's formulation of the Dirichlet problem (Definition 3.6):

Proposition 6.3. *Let Ω be a smooth bounded open set. Then, for every $\mu \in \mathcal{M}(\Omega)$, we have that u is a solution of the linear Dirichlet problem with density μ if and only if $u \in W_0^{1,1}(\Omega)$ and the equation $-\Delta u = \mu$ is satisfied in the sense of distributions in Ω: for every $\varphi \in C_c^\infty(\Omega)$,*

$$\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega \varphi \, \mathrm{d}\mu.$$

We begin with an elementary approximation procedure on the test functions in $C_0^\infty(\overline{\Omega})$:

Lemma 6.4. *Let Ω be a bounded open set. Then, for every nonnegative function $\zeta \in C_0^\infty(\overline{\Omega})$, there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of nonnegative functions in $C_c^\infty(\Omega)$ such that*

(I) *$(\varphi_n)_{n\in\mathbb{N}}$ converges uniformly to ζ in $\overline{\Omega}$,*
(II) *$(\nabla\varphi_n)_{n\in\mathbb{N}}$ is bounded in $\overline{\Omega}$ and converges pointwise to $\nabla\zeta$ in Ω.*

Proof of Lemma 6.4. Given a smooth function $H : \mathbb{R} \to \mathbb{R}$ vanishing in a neighborhood of 0, for every $n \in \mathbb{N}$ the function $\varphi_n : \overline{\Omega} \to \mathbb{R}$ defined by

$$\varphi_n = H(n\zeta)\zeta$$

belongs to $C_c^\infty(\Omega)$. The first assertion is then satisfied by choosing H such that

$$\lim_{t\to+\infty} H(t) = 1.$$

Concerning the convergence of the sequence of gradients, we first note that

$$\nabla\varphi_n = H(n\zeta)\nabla\zeta + [n\zeta H'(n\zeta)]\nabla\zeta.$$

Since ζ is nonnegative, we have $\nabla\zeta = 0$ in the set $\Omega \cap \{\zeta = 0\}$. Hence, the second assertion is satisfied by taking H such that

$$\lim_{t\to+\infty} tH'(t) = 0. \qquad \square$$

The previous lemma provides a similar approximation property for a signed function $\zeta \in C_0^\infty(\overline{\Omega})$. Indeed, we first write $\zeta = \zeta_1 - \zeta_2$ as a difference of two nonnegative functions in $C_0^\infty(\overline{\Omega})$. We then get an approximating sequence for ζ by applying the lemma separately to ζ_1 and ζ_2.

Proof of Proposition 6.3. If u is a solution of the Dirichlet problem, then the equation is satisfied in the sense of distributions, and by the Sobolev regularity property (Proposition 5.1) we have $u \in W_0^{1,1}(\Omega)$. Conversely, assume that $u \in W_0^{1,1}(\Omega)$ and that the equation is satisfied in the sense of distributions in Ω. On the one hand, since $u \in W_0^{1,1}(\Omega)$, for every $\zeta \in C_0^\infty(\overline{\Omega})$, we have

$$-\int_\Omega u\,\Delta\zeta = \int_\Omega \nabla u \cdot \nabla\zeta.$$

On the other hand, taking an approximating sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\Omega)$ satisfying properties (i) and (ii) from Lemma 6.4, for every $n \in \mathbb{N}$ we have

$$\int_\Omega \nabla u \cdot \nabla\varphi_n = \int_\Omega \varphi_n \,\mathrm{d}\mu.$$

As $n \to \infty$, for every $\zeta \in C_0^\infty(\overline{\Omega})$ we deduce from the dominated convergence theorem that

$$\int_\Omega \nabla u \cdot \nabla\zeta = \int_\Omega \zeta \,\mathrm{d}\mu.$$

Hence, u is a solution of the linear Dirichlet problem with density μ in the sense of Definition 3.1. □

Using the same argument, we obtain the equivalence between the notions of supersolutions for the Poisson equation and the Dirichlet problem for functions vanishing on the boundary in the sense of Sobolev functions:

Proposition 6.5. *Let Ω be a smooth bounded open set and let $\mu \in \mathcal{M}(\Omega)$. Take $u \in W_0^{1,1}(\Omega)$. The following assertions are equivalent*:

(i) $-\Delta u \geqslant \mu$ *in the sense of* $(C_0^\infty(\overline{\Omega}))'$,

(ii) $-\Delta u \geqslant \mu$ *in the sense of distributions in* Ω.

Proof. We only need to prove the reverse implication. Since $u \in W_0^{1,1}(\Omega)$, for every $\zeta \in C_0^\infty(\overline{\Omega})$ we have

$$-\int_\Omega u\,\Delta\zeta = \int_\Omega \nabla u \cdot \nabla\zeta.$$

Taking a nonnegative function ζ and a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\Omega)$ as in the approximation lemma above, for every $n \in \mathbb{N}$ we have

$$\int_\Omega \nabla u \cdot \nabla\varphi_n \geqslant \int_\Omega \varphi_n \,\mathrm{d}\mu.$$

Letting $n \to \infty$, by the dominated convergence theorem we get

$$\int_\Omega \nabla u \cdot \nabla\zeta = \lim_{n\to\infty}\int_\Omega \nabla u \cdot \nabla\varphi_n \geqslant \lim_{n\to\infty}\int_\Omega \varphi_n \,\mathrm{d}\mu = \int_\Omega \zeta \,\mathrm{d}\mu,$$

and the conclusion follows. □

The condition $u \in W_0^{1,1}(\Omega)$ is rather strong. A natural assumption – but more subtle to implement – is to assume that $u^- \in W_0^{1,1}(\Omega)$. The boundary information encoded in terms of test functions in $C_0^\infty(\overline{\Omega})$ can be also expressed as a limit of average integrals of u^- near the boundary (cf. Proposition 20.2):

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\{x \in \Omega : d(x, \partial\Omega) < \epsilon\}} u^- = 0,$$

without reference to Sobolev functions.

6.2 Variants of Kato's inequality

The classical weak maximum principle relies on the idea that the Laplacian of a smooth function is nonnegative in the minimum set of the function (cf. proof of Lemma 1.9). This leaves little hope that a similar interpretation holds for nonsmooth functions. In fact, this property has an elegant counterpart for potentials, and can be stated in terms of Kato's inequality, see Lemma A in [175]:

Proposition 6.6. *If $u \in L^1(\Omega)$ is such that $\Delta u \in L^1(\Omega)$, then*

$$\Delta u^+ \geqslant \chi_{\{u>0\}} \Delta u$$

in the sense of distributions in Ω.

The original motivation of Kato was to study properties of solutions of the Schrödinger equation which need not belong to the variational $W^{1,2}$ setting. Note that a twice differentiable function $u : \Omega \to \mathbb{R}$ satisfies, for every $x \in \Omega$,

$$\Delta u^+(x) = \begin{cases} \Delta u(x) & \text{if } u(x) > 0, \\ 0 & \text{if } u(x) < 0. \end{cases}$$

If $u(x) = 0$, then $\Delta u^+(x)$ need not exist in the classical sense. Since in this case x is a minimum point for the nonnegative function u^+, we could formally say that $\Delta u^+(x) \geqslant 0$. We thus obtain a formal pointwise statement of Kato's inequality, namely

$$\Delta u^+(x) \geqslant \chi_{\{u>0\}}(x) \Delta u(x).$$

Proof of Proposition 6.6. The first step of the proof of Kato's inequality relies on the observation that when u is smooth, then, for every smooth function $H : \mathbb{R} \to \mathbb{R}$, we have

$$\Delta H(u) = H'(u)\Delta u + H''(u)|\nabla u|^2,$$

and if in addition H is *convex*, then

$$\Delta H(u) \geqslant H'(u)\Delta u. \tag{6.1}$$

The next step consists in approximating $u \in L^1(\Omega)$ by smooth functions – for instance via convolution – in which case we may apply the inequality above. More precisely, given a nonnegative test function $\varphi \in C_c^\infty(\Omega)$, and given a sequence of mollifiers $(\rho_n)_{n\in\mathbb{N}}$ such that $\operatorname{supp}\varphi - \operatorname{supp}\rho_n \Subset \Omega$, we have

$$\Delta(\rho_n * u) = \rho_n * \Delta u \tag{6.2}$$

pointwise in $\operatorname{supp}\varphi$ (Lemma 2.22). Thus, integration by parts and inequality (6.1) for smooth functions yield

$$\int_\Omega H(\rho_n * u)\,\Delta\varphi = \int_\Omega \Delta H(\rho_n * u)\,\varphi \geqslant \int_\Omega H'(\rho_n * u)(\rho_n * \Delta u)\,\varphi. \tag{6.3}$$

Assuming in addition that the derivative H' is bounded in $\mathbb{R}$, as $n \to \infty$ we deduce from the dominated convergence theorem that

$$\int_\Omega H(u)\,\Delta\varphi \geqslant \int_\Omega H'(u)\Delta u\,\varphi. \tag{6.4}$$

In the last step, we approximate the function $\mathbb{R} \ni t \mapsto t^+$ by smooth convex functions. For this purpose, we take a sequence $(H_n)_{n\in\mathbb{N}}$ of smooth convex functions in $\mathbb{R}$ such that the sequence $(H_n')_{n\in\mathbb{N}}$ is uniformly bounded, and

(a) for every $t \in \mathbb{R}$, $\lim\limits_{n\to\infty} H_n(t) = t^+$,

(b) for every $t \leqslant 0$, $\lim\limits_{n\to\infty} H_n'(t) = 0$,

(c) for every $t > 0$, $\lim\limits_{n\to\infty} H_n'(t) = 1$.

Applying the integral inequality (6.4) with $H = H_n$, and letting $n \to \infty$, we get

$$\int_\Omega u^+\Delta\varphi \geqslant \int_\Omega \chi_{\{u>0\}}\Delta u\,\varphi.$$

This is the formulation of Kato's inequality in the sense of distributions in Ω. □

The assumption $\Delta u \in L^1(\Omega)$ is not invariant in the context of Kato's inequality:

Example 6.7. For every $a < c < b$, the function $u\colon (a,b) \to \mathbb{R}$ defined by $u(x) = x - c$ satisfies

$$\Delta u^+ = (u^+)'' = \delta_c$$

in the sense of distributions in (a,b). Hence, Δu^+ is not an L^1 function.

Note however that Δu^+ is always a locally finite measure, which follows from Schwartz's characterization of nonnegative distributions (Proposition 2.20). Indeed, by Kato's inequality, the distribution

$$T = \Delta u^+ - \chi_{\{u>0\}}\Delta u$$

is nonnegative, whence T is a locally finite measure on Ω. By linearity, we deduce that Δu^+ is also a locally finite measure on Ω.

Kato's inequality is usually applied to a solution of some equation, in which case the assumption $\Delta u \in L^1(\Omega)$ is probably enough. However, when dealing with subsolutions, or when the equation itself involves measure data, such an assumption on Δu becomes restrictive. In order to have a counterpart of Kato's inequality when Δu is a measure, one should first understand the meaning of the product $\chi_{\{u>0\}}\Delta u$, but this is a delicate issue. Indeed, if u and v are two functions which coincide almost everywhere in Ω, then Δu and Δv coincide as distributions, but $\chi_{\{u>0\}}\Delta u$ and $\chi_{\{v>0\}}\Delta v$ may be different.

We propose three ways of handling the product $\chi_{\{u>0\}}\Delta u$ in Kato's inequality when Δu need not be an L^1 function. The *first strategy* consists in eliminating the characteristic function $\chi_{\{u>0\}}$. For example, when u is smooth, we could write

$$\chi_{\{u>0\}}\Delta u \geqslant \min\{\Delta u, 0\}.$$

Proposition 6.8. *If $u \in L^1(\Omega)$ is such that $\Delta u \in \mathcal{M}(\Omega)$, then*

$$\Delta u^+ \geqslant \min\{\Delta u, 0\}$$

in the sense of distributions in Ω.

Proof. In the proof of Kato's inequality, we rewrite (6.2) in $\operatorname{supp}\varphi$ as

$$\Delta(\rho_n * u) = \rho_n * \Delta u \geqslant \rho_n * \min\{\Delta u, 0\}.$$

In particular, the function in the right-hand side is nonpositive. Assuming in addition that $0 \leqslant H' \leqslant 1$, the integral inequality (6.3) can now be replaced by

$$\int_\Omega H(\rho_n * u)\,\Delta\varphi \geqslant \int_\Omega (\rho_n * \min\{\Delta u, 0\})\,\varphi.$$

Thus, as $n \to \infty$, (6.4) is substituted by

$$\int_\Omega H(u)\,\Delta\varphi \geqslant \int_\Omega \varphi \min\{\Delta u, 0\}.$$

Using the same approximation argument with smooth convex functions, the conclusion follows. $\square$

The *second strategy* consists in replacing Δu by some summable function smaller than Δu. For example, when u is smooth and $\Delta u \geqslant f$ for some L^1 function f, we have

$$\chi_{\{u>0\}} \Delta u \geqslant \chi_{\{u>0\}} f.$$

Proposition 6.9. *Let $f \in L^1(\Omega)$. If $u \in L^1(\Omega)$ is such that*

$$\Delta u \geqslant f$$

in the sense of distributions in Ω, then

$$\Delta u^+ \geqslant \chi_{\{u>0\}} f$$

in the sense of distributions in Ω.

Proof. In the proof of Kato's inequality, we rewrite (6.2) in $\operatorname{supp} \varphi$ as

$$\Delta(\rho_n * u) = \rho_n * \Delta u \geqslant \rho_n * f.$$

Assuming in addition that H' is nonnegative, the integral inequality (6.3) becomes

$$\int_\Omega H(\rho_n * u)\Delta\varphi \geqslant \int_\Omega H'(\rho_n * u)(\rho_n * f)\,\varphi.$$

Thus, as $n \to \infty$, (6.4) is replaced by

$$\int_\Omega H(u)\Delta\varphi \geqslant \int_\Omega H'(u) f \varphi.$$

Using the same approximation argument with smooth convex functions, the conclusion follows. □

These two statements – or some combination of them – suffice for most purposes in applications. Besides, their proofs are as simple as the proof of the original Kato's inequality. As a drawback, we have to give up on part of the information carried by the measure Δu.

The *third strategy* consists in giving a meaning to $\chi_{\{u>0\}} \Delta u$ by choosing a suitable representative in some equivalence class of u. For this purpose, we adopt the precise representative $\hat{u}$ defined in the Lebesgue set $\mathcal{L}_u$ of u (Definition 8.3): for every $x \in \mathcal{L}_u$,

$$\boxed{\lim_{r \to 0} \fint_{B(x;r)} |u - \hat{u}(x)| = 0.}$$

where $\theta = \mu(\mathbb{R}^N)/(N-2)\sigma_N$. Since $\mu/\mu(\mathbb{R}^N)$ is a probability measure and g is a convex function, by Jensen's inequality we have

$$g(\mathcal{N}\mu(x)) \leqslant \int_{\mathbb{R}^N} g\Big(\frac{\theta}{|x-y|^{N-2}}\Big)\frac{\mathrm{d}\mu(y)}{\mu(\mathbb{R}^N)}. \tag{21.3}$$

By the monotonicity of g and the integration formula in polar coordinates, for every $R > 0$ we have

$$\begin{aligned}\int_{B(0;R)} g\Big(\frac{\theta}{|x-y|^{N-2}}\Big)\,\mathrm{d}x &\leqslant \int_{B(0;R)} g\Big(\frac{\theta}{|z|^{N-2}}\Big)\,\mathrm{d}z\\ &= \sigma_N \int_0^R g\Big(\frac{\theta}{r^{N-2}}\Big) r^{N-1}\,\mathrm{d}r.\end{aligned}$$

Making the change of variables $t = \theta/r^{N-2}$, we then get

$$\int_{B(0;R)} g\Big(\frac{\theta}{|x-y|^{N-2}}\Big)\,\mathrm{d}x \leqslant C_1 \int_{\frac{\theta}{R^{N-2}}}^{\infty} \frac{g(t)}{t^{\frac{N}{N-2}}}\frac{\mathrm{d}t}{t}.$$

Since the right-hand side is bounded from above independently of y, it follows from estimate (21.3) and Tonelli's theorem that $g(\mathcal{N}\mu) \in L^1(B(0;R))$. □

Proof of Proposition 21.2: "$\Longleftarrow$". Let $\mu_1 = \max\{\mu, 0\}$ and $\mu_2 = \min\{\mu, 0\}$. The function $\mathcal{N}\mu_1$ is nonnegative. Thus, by the sign condition, $g(\mathcal{N}\mu_1)$ is also nonnegative. By the previous lemma, we have $g(\mathcal{N}\mu_1) \in L^1(\Omega)$. Since $\mathcal{N}\mu_1$ is a nonnegative supersolution of the nonlinear Poisson equation with density μ, we deduce that $\mathcal{N}\mu_1$ is a supersolution of the nonlinear Dirichlet problem (Lemma 17.6).

By Property (21.2), we also have that $g(\mathcal{N}\mu_2) \in L^1(\Omega)$. Arguing as above, it follows that $\mathcal{N}\mu_2$ is a nonpositive subsolution of the nonlinear Dirichlet problem with density μ. Since g is nondecreasing, we may apply the method of sub- and supersolutions (Proposition 20.5) to conclude that the nonlinear Dirichlet problem with density μ has a solution u such that

$$\mathcal{N}\mu_2 \leqslant u \leqslant \mathcal{N}\mu_1. \qquad \square$$

To prove the direct implication of Proposition 21.2, we first study the behavior of spherical averages of potentials around a point.

Lemma 21.4. *Let $N \geqslant 3$. If $u \in L^1_{\mathrm{loc}}(\Omega)$ is such that $\Delta u \in \mathcal{M}_{\mathrm{loc}}(\Omega)$, then, for every $a \in \Omega$,*

$$\lim_{r\to 0} r^{N-2} \fint_{\partial B(a;r)} u\,\mathrm{d}\sigma = -\frac{1}{(N-2)\sigma_N}\Delta u(\{a\}).$$

Proof of Lemma 21.4. We temporarily assume that u is smooth. By the divergence theorem, we have (cf. proof of Lemma 1.5)

$$\frac{\mathrm{d}}{\mathrm{d}s}\fint_{\partial B(a;s)} u \,\mathrm{d}\sigma = \fint_{\partial B(a;s)} \frac{\partial u}{\partial n}\,\mathrm{d}\sigma = \frac{1}{\sigma_N s^{N-1}}\int_{B(a;s)} \Delta u.$$

By the fundamental theorem of calculus, for every $0 < r < \rho < d(a, \partial\Omega)$ we then get

$$\fint_{\partial B(a;r)} u \,\mathrm{d}\sigma = \fint_{\partial B(a;\rho)} u \,\mathrm{d}\sigma - \frac{1}{\sigma_N}\int_r^\rho \left(\int_{B(a;s)} \Delta u\right)\frac{\mathrm{d}s}{s^{N-1}}. \tag{21.4}$$

When u is merely $L^1_{\mathrm{loc}}(\Omega)$ and such that $\Delta u \in \mathcal{M}_{\mathrm{loc}}(\Omega)$, this identity holds for almost every r; this can be verified using an approximation argument (cf. Lemma 2.22). Since

$$\lim_{s\to 0}\int_{B(a;s)} \Delta u = \Delta u(\{a\}),$$

we have

$$\lim_{r\to 0}\,(N-2)r^{N-2}\int_r^\rho \left(\int_{B(a;s)} \Delta u\right)\frac{\mathrm{d}s}{s^{N-1}} = \Delta u(\{a\}).$$

Multiplying identity (21.4) by r^{N-2}, and letting $r \to 0$, the conclusion follows. □

Proof of Proposition 21.2: "$\Longrightarrow$". Let u be the solution of the nonlinear Dirichlet problem with density δ_a, for some $a \in \Omega$. By the integration formula in polar coordinates, for every $0 < \epsilon < d(a, \partial\Omega)$ we have

$$\int_{B(a;\epsilon)} g(u) = \int_0^\epsilon \left(\int_{\partial B(a;r)} g(u)\,\mathrm{d}\sigma\right)\mathrm{d}r = \sigma_N \int_0^\epsilon r^{N-1}\left(\fint_{\partial B(a;r)} g(u)\,\mathrm{d}\sigma\right)\mathrm{d}r.$$

Since g is convex, by Jensen's inequality we have

$$\fint_{\partial B(a;r)} g(u)\,\mathrm{d}\sigma \geqslant g\left(\fint_{\partial B(a;r)} u \,\mathrm{d}\sigma\right).$$

Given $0 < \theta < 1/(N-2)\sigma_N$, the previous lemma shows that there exists $\epsilon > 0$ such that, for almost every $0 < r \leqslant \epsilon$,

$$\fint_{\partial B(a;r)} u \,\mathrm{d}\sigma \geqslant \frac{\theta}{r^{N-2}}.$$

Since g is nondecreasing, we deduce that

$$\int_{B(a;\epsilon)} g(u) \geqslant \sigma_N \int_0^\epsilon r^{N-1} g\left(\frac{\theta}{r^{N-2}}\right)\mathrm{d}r.$$

Making the change of variables $t = \theta/r^{N-2}$, we have the conclusion. □

Chapter 22
The Schrödinger operator

> "The customary method does not seem to work owing to the high singularity of the potential."
>
> Tosio Kato

The Schrödinger operator $-\Delta + V$ is associated to a force field of the form $-\nabla V$. We establish a strong maximum principle for nonnegative smooth functions satisfying

$$-\Delta u + Vu \geqslant 0 \quad \text{in } \Omega,$$

where the potential V merely belongs to the Lebesgue space $L^p(\Omega)$ for some $1 \leqslant p \leqslant +\infty$. The proof relies on the existence of solutions of the Dirichlet problem for the Schrödinger operator involving measures.

22.1 Strong maximum principle

The classical strong maximum principle for the Laplacian (Lemma 1.11) implies that if Ω is a connected open set and if $u\colon \Omega \to \mathbb{R}$ is a nonnegative smooth function such that

$$-\Delta u \geqslant 0 \quad \text{in } \Omega,$$

then either $u = 0$ in Ω or $u > 0$ in Ω. Such a property is also satisfied by Schrödinger operators with bounded potentials:

Proposition 22.1. *Let Ω be a connected open set, and let $V \in L^\infty(\Omega)$. If $u\colon \Omega \to \mathbb{R}$ is a nonnegative smooth function such that*

$$-\Delta u + Vu \geqslant 0 \quad \textit{in } \Omega,$$

and if there exists $a \in \Omega$ such that $u(a) = 0$, then $u = 0$ in Ω.

Proof. Given $\lambda > 0$, consider the nonnegative function

$$U\colon \Omega \times \Big(-\frac{\pi}{2\lambda}, \frac{\pi}{2\lambda}\Big) \longrightarrow \mathbb{R}$$

defined by

$$U(x, s) = u(x) \cos{(\lambda s)}.$$

Computing the Laplacian of U with respect to the variable (x, s) we get

$$\Delta U(x,s) = [\Delta u(x) - \lambda^2 u(x)] \cos(\lambda s).$$

Since $\cos \lambda s \geqslant 0$, by the assumption on u we get

$$\Delta U(x,s) \leqslant [V(x) - \lambda^2] U(x,s).$$

Choosing $\lambda \geqslant \|V\|_{L^\infty(\Omega)}^{1/2}$, the function U is superharmonic. Now, if $u(a) = 0$ for some $a \in \Omega$, then $U(a, 0) = 0$, and it follows from the strong maximum principle for superharmonic functions (Lemma 1.11) that U is identically zero. This implies the conclusion for the function u. □

By the Harnack inequality (see Theorem 5 in [308], Corollaire 8.1 in [316], or Theorem 5.2 in [326]) based on Moser's iteration technique [257], the same conclusion remains true for potentials $V \in L^p(\Omega)$ for some exponent $p > \frac{N}{2}$. Below this threshold, nonnegative supersolutions of the Schrödinger operator may vanish without being identically zero: the function $u \colon B(0;1) \to \mathbb{R}$ defined by $u(x) = |x|^2$ satisfies the equation

$$-\Delta u + Vu = 0 \quad \text{in } B(0;1)$$

with $V(x) = 2N/|x|^2$. In this case, we have $V \in L^p(B(0;1))$ for every $1 \leqslant p < \frac{N}{2}$, but $V \notin L^{N/2}(B(0;1))$.

When the supersolution u vanishes on a sufficiently large set, one would still hope to conclude that $u = 0$ in Ω for badly behaved potentials like $V \in L^1(\Omega)$, see Theorem C.1 in [24], Theorem 5.2 in [87], or Theorem in [327]. Ancona beautifully identified the role played by the $W^{1,2}$ capacity to detect the size of the set $\{u = 0\}$, see Theorem 9 in [14]:

Proposition 22.2. *Let Ω be a connected open set, and let $V \in L^1(\Omega)$. If $u \colon \Omega \to \mathbb{R}$ is a nonnegative smooth function such that*

$$-\Delta u + Vu \geqslant 0 \quad \text{in } \Omega,$$

and if u vanishes in a compact subset of positive $W^{1,2}$ capacity, then $u = 0$ in Ω.

Ancona's proof relies on tools from potential theory. We present an alternative strategy based on the following estimate, see [64] and [327]:

Lemma 22.3. *Let $V \in L^1(\Omega)$, and let $u \colon \Omega \to \mathbb{R}$ be a nonnegative smooth function. If*

$$-\Delta u + Vu \geqslant 0 \quad \text{in } \Omega,$$

then, for every $\varphi \in C_c^\infty(\Omega)$, we have

$$\int_\Omega |\nabla \log(1+u)|^2 \varphi^2 \leqslant C \int_\Omega (V^+ \varphi^2 + |\nabla \varphi|^2).$$

Proof of Lemma 22.3. Given $\varphi \in C_c^\infty(\Omega)$, we multiply the differential inequality by $\varphi^2/(1+u)$ to get

$$\Delta u \, \frac{\varphi^2}{1+u} \leqslant \frac{Vu}{1+u}\varphi^2 \leqslant V^+\varphi^2.$$

Note that

$$\operatorname{div}\Big(\nabla u \, \frac{\varphi^2}{1+u}\Big) = \Delta u \, \frac{\varphi^2}{1+u} + \nabla u \cdot \nabla\varphi \frac{2\varphi}{1+u} - \frac{|\nabla u|^2}{(1+u)^2}\varphi^2.$$

Thus,

$$\frac{|\nabla u|^2}{(1+u)^2}\varphi^2 \leqslant V^+\varphi^2 + \nabla u \cdot \nabla\varphi \frac{2\varphi}{1+u} - \operatorname{div}\Big(\nabla u \, \frac{\varphi^2}{1+u}\Big). \tag{22.1}$$

On the other hand, for every $\epsilon > 0$ we have (Exercise 4.14)

$$\nabla u \cdot \nabla\varphi \frac{2\varphi}{1+u} \leqslant \epsilon \frac{|\nabla u|^2}{(1+u)^2}\varphi^2 + \frac{1}{\epsilon}|\nabla\varphi|^2.$$

Inserting this inequality in (22.1), we get

$$(1-\epsilon)\frac{|\nabla u|^2}{(1+u)^2}\varphi^2 \leqslant V^+\varphi^2 + \frac{1}{\epsilon}|\nabla\varphi|^2 - \operatorname{div}\Big(\nabla u \, \frac{\varphi^2}{1+u}\Big).$$

Integrating both sides over Ω, it follows from the divergence theorem that

$$(1-\epsilon)\int_\Omega \frac{|\nabla u|^2}{(1+u)^2}\varphi^2 \leqslant \int_\Omega V^+\varphi^2 + \frac{1}{\epsilon}\int_\Omega |\nabla\varphi|^2.$$

We deduce the estimate by taking any fixed $0 < \epsilon < 1$. □

We need the following variant of the Poincaré inequality for functions vanishing on a set of positive $W^{1,2}$ capacity:

Lemma 22.4. *Let Ω be a connected smooth bounded open set, and let $K \subset \Omega$ be a compact set. If $\operatorname{cap}_{W^{1,2}}(K) > 0$, then there exists $C > 0$ such that, for every function $\psi \in C^\infty(\overline{\Omega})$ vanishing in K, we have*

$$\|\psi\|_{L^2(\Omega)} \leqslant C\|\nabla\psi\|_{L^2(\Omega)}.$$

Proof of Lemma 22.4. Arguing by contradiction, if the inequality is not true, then there exists a sequence $(\psi_n)_{n\in\mathbb{N}_*}$ in $C^\infty(\overline{\Omega})$ such that, for every $n \in \mathbb{N}_*$,

(*a*) $\psi_n = 0$ in K,

(*b*) $\|\psi_n\|_{L^2(\Omega)} = 1$,

(*c*) $\|\nabla\psi_n\|_{L^2(\Omega)} \leqslant \frac{1}{n}$.

Thanks to the smoothness of Ω, we may extend ψ_n as a function supported in a ball $B(0; R) \Supset \Omega$, with uniform control of the $W^{1,2}$ norm. Thus, by the Rellich–Kondrashov compactness theorem (Proposition 4.8), there exists a subsequence $(\psi_{n_k})_{k\in\mathbb{N}}$ converging strongly in $L^2(\Omega)$ to some function u such that

$$\|u\|_{L^2(\Omega)} = 1.$$

Since $(\nabla\psi_{n_k})_{k\in\mathbb{N}}$ converges to 0 in $L^2(\Omega; \mathbb{R}^N)$, we deduce that $u \in W^{1,2}(\Omega)$ and $\|\nabla u\|_{L^2(\Omega)} = 0$. By the connectedness of Ω, we then have $u = \alpha$ almost everywhere in Ω for some $\alpha \in \mathbb{R}$. Since $|\alpha|\,|\Omega|^{\frac{1}{2}} = 1$, we have that $\alpha \neq 0$.

Given a nonnegative function $\varphi \in C_c^\infty(\Omega)$ such that $\varphi > 1$ in K, for each $k \in \mathbb{N}$ the function $(1 - \psi_{n_k}/\alpha)\,\varphi$ is admissible in the definition of the $W^{1,2}$ capacity of K, whence

$$\operatorname{cap}_{W^{1,2}}(K) \leqslant \|(1 - \psi_{n_k}/\alpha)\,\varphi\|_{W^{1,2}(\Omega)}^2.$$

As $k \to \infty$, the quantity in the right-hand side converges to zero, and we deduce that $\operatorname{cap}_{W^{1,2}}(K) = 0$. This is a contradiction. □

Proof of Proposition 22.2. For every $\delta > 0$, the function u/δ satisfies the assumptions of Lemma 22.3. Thus, for every smooth bounded open subset $\omega \Subset \Omega$, there exists a constant $C_1 > 0$, independent of $\delta > 0$, such that

$$\int_\omega \left|\nabla \log\left(1 + \frac{u}{\delta}\right)\right|^2 \leqslant C_1.$$

Since the function $\log(1 + u/\delta)$ vanishes in a compact subset of positive $W^{1,2}$ capacity in Ω, we may choose a connected smooth bounded open set $\omega \Subset \Omega$ having the same property. Thus, by the Poincaré inequality above, we have

$$\int_\omega \left|\log\left(1 + \frac{u}{\delta}\right)\right|^2 \leqslant C^2 \int_\omega \left|\nabla \log\left(1 + \frac{u}{\delta}\right)\right|^2 \leqslant C_2.$$

By the Chebyshev inequality and the monotonicity of the logarithm, for every $t > 0$,

$$|\omega \cap \{u > t\}|\left|\log\left(1 + \frac{t}{\delta}\right)\right|^2 \leqslant C_2.$$

Letting $\delta \to 0$, we deduce that $|\omega \cap \{u > t\}| = 0$. Thus, $0 \leqslant u \leqslant t$ in ω, for every $t > 0$, whence $u = 0$ in ω. □

It follows from Proposition 22.2 that either $u = 0$ in Ω, or the set $\{u = 0\}$ has Hausdorff dimension at most $N - 2$ (cf. Proposition 10.4). The $W^{1,2}$ capacity gives a more precise information in the sense that the following converse of the strong maximum principle holds, see Proposition 6.3 in [270]: for any compact set $K \subset \mathbb{R}^N$